Planes, Trains, and Buses: Seat Allocation Problem*

Mustafa Oğuz Afacan† Ayşe Dur‡ Umut Dur§

April 5, 2022

Abstract

We study a seat allocation problem in public transportation. Motivated by real-life practice, we consider gender-based restrictions and no restriction cases. Under the former, no pair of passengers of different gender can be seated next to each other, while under the latter, there are no restrictions. We first show that the commonly used procedure suffers from serious handicaps. We then introduce a new mechanism that avoids all these deficiencies, while also satisfying some other desirable properties. We also show that our proposal is the only stable mechanism. We run simulations to quantify the gain from replacing the current procedure with our proposal.

*We would like to thank Azer Abizade for his comments.

†Faculty of Arts and Social Sciences, Sabanci University, Istanbul, Turkey; email: mafacan@sabanciuniv.edu

‡Department of Economics, North Carolina State University, Raleigh, NC, USA; email: azkabukc@ncsu.edu

§Department of Economics, North Carolina State University, Raleigh, NC, USA; email: umutdur@gmail.com

1 Introduction

In many countries, seat allocation in public transportation is subject to gender-based restrictions. For instance, in India, Delhi Metro reserves the first coach of every train and a few seats in each compartment for women. Train services in Mumbai and Kolkata, have ladies-only compartments.[1] Air India, the flagship carrier of India, reserves six seats in the economy class for women who travel alone.[2] Similar practices of gender-based reserved seating is being used in Japan, Indonesia, Germany, and Malaysia.[3]

A slightly different gender-based reserved seating used in the other countries is that inter-city buses and trains do not allow a seat next to a woman (or man) to be taken by a man (or woman) when these passengers are traveling alone. For instance, in Turkey, ticket sales and reservations for intercity buses and trains are done online and each passenger can reserve any seat s/he wants—as long as the next seat has not been reserved by a passenger of different gender. Moreover, a passenger cannot be forced to move to another seat after reserving a seat. Hence, current restrictions may lead to a situation in which only one seat is readily reserved by a passenger, and the seat next to it cannot be reserved by another of a different gender. This may cause a large number of vacant seats and potential fairness violations in the sense that a passenger may envy someone else even though the former makes a reservation earlier. Figure 1 illustrates this situation: There are only three seats left in the economy class and every vacant seat is next to a woman. In this case, male passengers will not be allowed to buy a ticket in the economy class.

In Turkey, passengers have complained about the issues raised by this restriction in various online platforms. For instance, a passenger who cannot purchase a ticket due to this restriction expressed his complaint on www.sikayetimvar.com.[4]

---

[1]See https://citizenmatters.in/reserved-seats-for-women-safety-on-public-transport-4901.

[2]https://www.businessinsider.com/air-india-women-female-only-seats-2017-1.

[3]See http://news.bbc.co.uk/2/hi/asia-pacific/1055599.stm, https://www.bbc.com/news/world-asia-pacific-11028078, https://www.dw.com/en/opinion-divided-over-women-only-train-compartments-on-eastern-german-route/a-19192346, and https://www.nbcnews.com/id/wbna40468372#.U0578CiUc_g.

[4]www.sikayetimvar.com is a website where people can express their complaints about the companies and brands in Turkey. See https://www.sikayetvar.com/tcdd/tcdd-hizli-tren-rezervasyon-sorunu.

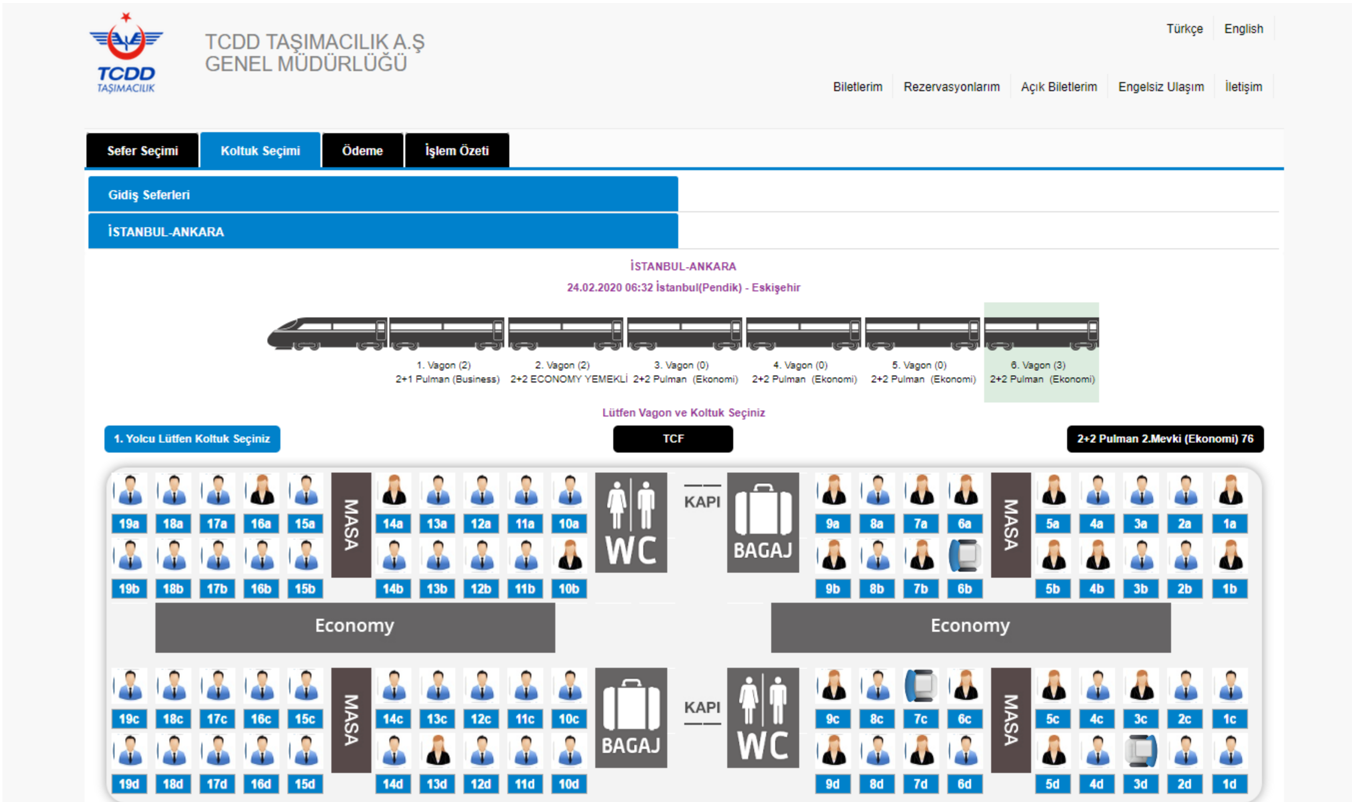

Figure 1: Train Seat Selection in Turkey

"... There are about 30 empty seats on the train, but I cannot reserve because there is a woman next to every vacant seat, and the system does not allow me to reserve one. There cannot be such a system where there is a space on the train, but I can't get it. I have been calling the help center but nobody helps, isn't it unfair?"

Another frustrated passenger expressed his experience as follows:

"For my trip on Friday, I have been trying to buy tickets since Wednesday. It was full for several days. I just saw 11 vacancies on a train. ...but since there are 11 women sitting alone, I cannot buy tickets right now. I wonder when officials will stop this nonsense."

The complaints do not only come from the male passengers. A female passenger made the following comment:

> *"Although there are 20 vacancies, when I move to the next stage of online reservation, I cannot find a place as a woman because there are always men sitting next to the vacant seats, the system prevents me from choosing a seat."*[5]

An easy solution to solve the problems raised due to this gender-based seating restriction is removing it. However, there are passengers who support this restriction since they feel more secure.[6]

Apart from removing the gender-based seating restriction, passengers also provide suggestions for solving these problems caused due to this restriction:

> *"A solution can be found. When vacant seats are next to female (male) passengers on the train and a man (woman) is trying to buy a ticket, one female (male) passenger can be moved randomly next to another woman (man)."*[7]

Motivated by the issues experienced under the gender-based seating restrictions in Turkey, we formulate the problem and aim to come up with a desirable seat assignment mechanism. More specifically, we consider a problem with a set of seats to be allocated to passengers. Seats are grouped in rows, each including either two or three seats.[8] Passengers have preferences over the seats and are prioritized based on their reservation time. Each passenger is either male or female. Gender-based seating restrictions require that no two passengers of a different gender are seated in adjacent seats.[9]

We introduce a stability notion ensuring that (i) there is no way of giving a seat to an unseated passenger without displacing some higher priority passenger, and (ii) no passenger can propose an alternative seat assignment under which s/he is better off without harming

[5] See https://www.sikayetvar.com/tcdd/erkek.

[6] See https://www.bbc.com/turkce/haberler-dunya-41662919.

[7] See https://eksisozluk.com/yhtde-bayan-yani-uygulamasi–4524861?p=14

[8] We choose this setting to cover the seating schemes in buses, trains, and planes.

[9] In a row with three seats, a male and a female passenger can be seated as long as the middle seat is left empty.

anyone with a higher priority and causing a new unseated passenger. Under the gender-based seating restriction, we first show that the common practice of first-come-first-served seating, where passengers choose their seats by their ticket purchase time, is far from desirable. We call this mechanism "*Myopic Serial Dictatorship*" (MSD) and show that it is not stable, strategy-proof,[10] efficient, or maximal in the sense that its outcome can be improved in terms of the number of assigned passengers.

Given the serious deficiencies of MSD, we next introduce a mechanism, called "*Adaptive Serial Dictatorship*" (ASD). In ASD, passengers choose their favorite seat one by one in order of their reservation time using carefully defined choice sets. The choice set construction helps ASD achieve desirable properties. We show that ASD is stable, efficient, and strategy-proof; however, it is not maximal. Nonetheless, its lack of maximality is not specific to ASD, as there is a general tension between stability and maximality. Fortunately, it is constrained maximal in the sense that the ASD outcome can never be improved in terms of the number of agents assigned by a stable matching. We also show that MSD is not even constrained maximal.

Another desirable property of a mechanism is that passengers should not be penalized for an early arrival time. To address this, we introduce a respecting improvement notion and show that ASD respects improvements. Hence, it incentivizes passengers for to reserve earlier. We also obtain a characterization of ASD: it is the unique stable mechanism.

Not all seat allocation problems in transportation are subject to gender-based seating restrictions. For instance, Southwest Airlines does not assign seats a priori. Instead, passengers select their seats after they enter the plane without any restriction. We study such restriction-free seat assignments as well. For this purpose, we straightforwardly adapt MSD and ASD to this case. While MSD becomes maximal, its negative properties above continue holding. ASD, on the other hand, maintains its positive properties as it becomes maximal.

Finally, by using simulations, we measure the gain from replacing MSD with ASD in

[10] A mechanism is strategy-proof if no agent ever has an incentive to misreport his/her preferences.

terms of the number of assigned passengers and elimination of stability violations.

The applicability of the theoretical framework is not limited to the seat allocation problem. For instance, during the COVID-19 pandemic, in order to maintain social distancing, many airlines blocked the middle seat in the planes with 3+3 seating configuration.[11] Moreover, airlines are currently considering implementing vaccination passports.[12] Our model can be applied to allocate seats under vaccination passport restrictions where a vaccinated passenger can be seated next to any other. However, a pair of unvaccinated passengers cannot be seated next to each other.[13] Our proposed mechanism can be modified to find the best solution under this application. We also discuss some model extensions, including couples and weak preferences, in the Discussion section.

Matching theory has been applied to many real-life markets, including school choice (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu et al., 2009; Kesten, 2010), organ exchange (Roth et al., 2004; Ergin et al., 2017), refugee resettlement (Trapp et al., 2020), and cadet-branch matching (Sönmez and Switzer, 2012). To the best of our knowledge, this paper is the first application of matching theory to the seat allocation problem in public transportation. While there is no closely related paper, our model exhibits externalities,[14] and hence it is generally related to the matching with externalities literature. Sasaki and Toda (1996) define a stability notion where the blocking agents consider all possible reactions from others. They show a stable matching always exists whenever agents always consider the worst-case scenarios. Pycia and Yenmez (2019) introduce substitutes and irrelevance of rejected contracts conditions and obtain the existence of a stable matching under these conditions. Fisher and Hafalir (2016) consider a one-to-one matching problem where agents ignore the effects of the externalities of their actions. They find some conditions

[11] See https://www.cnn.com/2021/04/14/health/airplane-seating-covid-risk-cdc-study-wellness/index.html, https://news.delta.com/delta-extends-middle-seat-blocking-through-april-2021-only-us-airline-continue-providing-more-space, https://www.forbes.com/sites/advisor/2020/12/07/master-list-of-us-airline-seating-and-mask-covid-19-policies/?sh=4aee98f11bb4.

[12] See https://www.nbcnews.com/business/travel/next-frontier-air-travel-digital-passports-proof-vaccination-n1261338.

[13] The airline may require extra fees from these passengers if their next seat needs to be blocked.

[14] Passengers' welfare is affected by the others' seat assignments.

under which a stable matching always exists. Hafalir (2008) formulates a marriage problem with endogenous beliefs as to others' reactions to a blocking pair. He comes up with a particular belief formation with which the existence of a stable matching is guaranteed.

There is a body of literature on seat allocation problem, but from a completely different perspective from ours. This literature mainly studies how to increase revenue from seat sales. Based on consumer behavior, Yuan and Nie (2020), investigate how seat-grouping in trains should be done in China to increase revenue. Sawaki (1989) considers a price discrimination model and finds the optimal number of seats that should be sold for a low fare to maximize the expected revenue. Freisleben and Gleichmann (1993) study overbooking predictions to decrease the empty seats in flights.

2 Model

In this section, we first introduce the **seat allocation problem**. Then, we provide the axioms used in our analysis.

2.1 Seat Allocation Problem

Let $(N, S, \rhd, \succ)$ be a seat-allocation problem described below.

- N and S are the non-empty sets of **agents** (passengers) and **seats**, respectively.

- Each agent is either **male or female**.[15] Let N^m and N^f be the sets of male and female agents, respectively.

- $\succ$ is the **priority ordering** over the agents such that the earlier an agent makes a reservation, the higher priority s/he has. For any pair of agents i, j, $i \succ j$ means that agent i comes before agent j in the ordering, and thereby has a priority over agent j. Let $N = \{i_1, .., i_n\}$ be the enumeration of the agents such that for each $k < k'$,

[15] These types may differ in other applications.

$i_k \succ i_{k'}$.[16] Let $U(i_k) = \{i_{k'} : k' < k\}$, that is, the set of agents who come earlier than i_k in the ordering.

- Each agent i has a **strict ranking** $\rhd_i$ over S. Let $\rhd = (\rhd_i)_{i \in N}$ be the ranking profile.

Seats are grouped in rows consisting of either 2 or 3 seats. Let $r \geq 1$ be the total number of rows. We write σ_s for the type of seat s. In the case of 2-seat rows, $\sigma_s \in \{1, .., r\} \times \{W, A\}$; and otherwise, $\sigma_s \in \{1, .., r\} \times \{W, M, A\}$. Its first and second components denote the row and side of seat s where W, M, A stand for window, middle, and aisle sides, respectively.

We say that a seat is **adjacent** of another if they are in the same row and next to each other. For instance, in the case of 2-seat rows, seats in the same row are adjacent to each other. In the case of 3-seat rows, each seat-pair in a row except the window-aisle one is adjacent to each other. Let τ_s be the seats that are in the same row as seat s, including seat s. Note that $|\tau_s|$ gives us the number of seats each row contains.

Regarding the agents' seat orderings, we assume that in the case of 3-seat rows, each agent ranks the middle seat below the two other seats in the same row.

Assumption 1. *In the case of* 3*-seat rows, for each agent* i *and pair of seats* $s, s' \in \tau_{s'}$ *where* $\sigma_{s'} = (r, M)$, $s \rhd_i s'$.

A **matching** μ is an assignment of seats to agents such that each agent receives at most one seat, and no seat is assigned to more than one agent. For any $k \in N \cup S$, we write μ_k for the assignment of k under μ. We write $\mu_k = \emptyset$ if agent (seat) k does not receive a seat (is not assigned to an agent). Let $\mu_S = \{i \in N : \mu_i \neq \emptyset\}$. Under gender-based seating restrictions, a matching μ is **feasible** if there do not exist $i \in N^m$ and $j \in N^f$ such that μ_i and μ_j are adjacent seats. When there is no gender-based seating restriction, any matching is feasible. In the rest of the paper, we only consider feasible matchings, and for ease of exposition, we just refer to them as matching.

[16] In the rest of the paper, we use this ordering unless otherwise stated.

An agent's well-being at a matching depends not only on his seat assignment per se, but also the availability statutes of the other seats in the same row. In other words, the problem exhibits externalities, hence agents' ordering over the seats are not capable of representing their preferences over the matchings. Here, we assume a particular class of externalities, where each agent always prefers having the other seats in his assigned row be empty.

Assumption 2. *An agent i prefers matching μ to ν if either*

(i) $\mu_i \in S$ *and* $\nu_i = \emptyset$, *or*

(ii) $|\{s'' \in \tau_{\mu_i} : \mu_{s''} = \emptyset\}| > |\{s'' \in \tau_{\nu_i} : \nu_{s''} = \emptyset\}|$, *or*

(iii) at least one adjacent seat of μ_i *is empty while none of the adjacent seats of* ν_i *are empty, or*

(iv) none of the first three cases holds and $\mu_i \rhd_i \nu_i$.

In other words, the first condition assures that each agent always prefers receiving a seat. The second condition, on the other hand, indicates that each agent always prefers having more empty seats in his assigned row. In the case of the same number of empty seats in his row, he prefers having the adjacent seats empty. If none of these holds, only then do his preferences come from his ordering over the seats. Note that we do not impose any preferential supposition over pairs of matchings where an agent receives the same middle seat in both cases, but a different adjacent seat is empty in each case. Agents can be indifferent or have strict preferences between such two matchings.[17]

Let R_i denote the agent i's preferences over matchings. We write P_i for its strict part. Two notes are in order: (i) each $\rhd_i$ induces different preferences, and (ii) agents are all indifferent between matchings where their seats and the availability statutes of the other seats in their rows are the same. That is, they do not have preferences over the agents

[17]This case does not matter in our solution, as, in line with Assumption 1, it always leaves the middle seats empty to the extent possible.

seated in the same row. In the rest of the paper, we fix all the primitives except the agents' ranking over the seats and denote the problem by $\rhd$.

2.2 Axioms

Next, we define the axioms used in our analysis. A matching μ is **stable** if, for each agent i_k, (a) whenever $\mu_{i_k} = \emptyset$, there is no matching μ' where $\mu'_{i_k} \neq \emptyset$ and $U(i_k) \cap \mu_S \subseteq U(i_k) \cap \mu'_S$, and (b) whenever $\mu_{i_k} \neq \emptyset$, there is no matching μ' where $\mu'_S = \mu_S$, $\mu' \ P_{i_k} \ \mu$, and for each $j \in U(i_k)$, $\mu' \ R_j \ \mu$. Our stability notion is different from its usual definition in standard object assignment problems because of the externalities in the seat allocation problem. Less formally, condition (a) ensures that no seat is wasted (see Remark 1 below for details) and no agent is unassigned for the sake of a lower priority agent. Condition (b), on the other hand, imposes that no agent can be better off without harming a higher priority one or causing someone to be unseated.

Remark 1. *The stability of a matching μ implies that for each agent i with $\mu_i = \emptyset$, there exists no matching μ' where $\mu'_S = \mu_S \cup \{i\}$. That is, no agent can receive a seat without creating a newly unassigned agent. In other words, no seat is wasted, which is a property known as non-wastefulness, implied by stability.*

Matching μ **Pareto dominates** μ' if for each agent $i \in N$, $\mu \ R_i \ \mu'$, where this strictly holds for some agent. Matching μ is **efficient** if it is not Pareto dominated by another matching. A matching μ is **maximal** if there does not exist another matching ν such that $|\mu_S| < |\nu_S|$, i.e., no matching allocates more seats than μ.

A **mechanism** ψ is a systematic way to produce a matching for each problem $\rhd$. We write $\psi(\rhd)$ to denote the outcome of ψ at problem $\rhd$. Mechanism ψ is < *stable, efficient, maximal* > if $\psi(\rhd)$ is <stable, efficient, maximal > for each problem $\rhd$. Mechanism ψ is **strategy-proof** if there is no problem $\rhd$ and agent i with $\rhd'_i$ such that $\psi(\rhd'_i, \rhd_{-i}) \ P_i \ \psi(\rhd)$.[18]

[18] $\rhd_{-i}$ is the ranking profile of all the agents, except agent i, over the seats.

3 Results

In what follows, we provide the results in two sections: **Restricted** and **Unrestricted** cases. In the former, no pair of agents of different genders can be seated next to next. We impose it as a feasibility requirement over matchings and only consider those satisfying it. There is no change in the axioms except, in their definitions, we only consider matchings that satisfy this restriction. In the unrestricted case, on the other hand, each agent can be seated next to any one irrespective of their genders. Hence, we do not have any feasibility requirement over matchings. The following results holds for both restricted and unrestricted cases.

Proposition 1. *Let μ be a stable matching in problem $\triangleright$. Then, μ is efficient.*

We provide all the proofs in the Appendix.

3.1 Restricted Case

In this section, we assume that no pair of agents of different genders can be seated next to each other. We start our analysis by first defining a mechanism, called *Myopic Serial Dictatorship*, that mimics the common current practice in seat allocations.

3.1.1 Myopic Serial Dictatorship (MSD)

MSD is based on serial dictatorship in which each agent selects his most preferred seat one by one following the ordering. As this selection is done without knowing the selections' of the agents coming later in the ordering, the outcome may be undesirable in many aspects, as we will discuss. We formally define the MSD mechanism below.

Myopic Serial Dictatorship:

By following the agent-ordering, the MSD mechanism selects its outcome through the following steps. For $k \in \{1, \ldots, n\}$

Step k**.** Let us consider the seats in the rows including the highest number of empty seats. Among these seats, agent i_k selects the best ranked one (with respect to $\triangleright_{i_k}$) whose adjacent was not taken by an agent of different gender in a previous step. If such a seat does not exist, then i_k stays unassigned.

MSD terminates by the end of Step n, that is, when all agents are processed. We first illustrate how MSD works via a simple example.

Example 1. *Let* $N^m = \{m_1, m_2, m_3, m_4\}$, $N^f = \{f_1\}$, $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$, $\tau_{s_1} = s_2$, $\tau_{s_3} = s_4$, $\tau_{s_5} = s_6$, *and* $m_1 \succ m_2 \succ m_3 \succ m_4 \succ f_1$. *Agents' strict rankings over* S *are:*

$$s_1 \triangleright_{m_1} s_2 \triangleright_{m_1} s_3 \triangleright_{m_1} s_4 \triangleright_{m_1} s_5 \triangleright_{m_1} s_6$$

$$s_3 \triangleright_{m_2} s_2 \triangleright_{m_2} s_1 \triangleright_{m_2} s_4 \triangleright_{m_2} s_5 \triangleright_{m_2} s_6$$

$$s_3 \triangleright_{m_3} s_1 \triangleright_{m_3} s_2 \triangleright_{m_3} s_4 \triangleright_{m_3} s_5 \triangleright_{m_3} s_6$$

$$s_2 \triangleright_{m_4} s_1 \triangleright_{m_4} s_3 \triangleright_{m_4} s_4 \triangleright_{m_4} s_5 \triangleright_{m_4} s_6$$

$$s_2 \triangleright_{f_1} s_1 \triangleright_{f_1} s_3 \triangleright_{f_1} s_4 \triangleright_{f_1} s_5 \triangleright_{f_1} s_6$$

Let us run MSD *in the problem.*

Step 1**.** *Since all the seats are empty, agent* m_1 *selects* s_1, *which is the best ranked seat with respect to* $\triangleright_{m_1}$.

Step 2**.** *Agent* m_2 *selects* s_3, *which is the best ranked seat with respect to* $\triangleright_{m_2}$ *in the rows with the highest number of empty seats.*

Step 3**.** *Agent* m_3 *selects* s_5, *which is the best ranked seat with respect to* $\triangleright_{m_3}$ *in the rows with the highest number of empty seats.*

Step 4**.** *Since a seat in each row has already been taken in the previous steps, agent* m_4 *selects* s_2, *which is the best ranked seat with respect to* $\triangleright_{m_4}$.

Step 5**.** *Since each row contains a man, agent* f_1 *cannot be seated.*

Hence, MSD *selects matching* μ *where* $\mu_{m_1} = s_1$, $\mu_{m_2} = s_3$, $\mu_{m_3} = s_5$, $\mu_{m_4} = s_2$, *and* $\mu_{f_1} = \emptyset$.

Below, we show that MSD fails to satisfy desirable properties.

Proposition 2. *In the restricted case, MSD is not stable, or maximal, or efficient, or strategy-proof.*

Given the negative results presented in Proposition 2, our objective is to introduce a mechanism satisfying desirable properties to the extend that possible. Unfortunately, Proposition 3 shows that, in the restricted case, stability and maximality are incompatible.

Proposition 3. *In the restricted case, there does not always exists a stable and maximal matching.*

Proposition 3 implies that there does not exist a stable mechanism that is also maximal. Given the incompatibility between stability and maximality, we focus on the maximality among stable matchings. We say that a matching μ is **constrained maximal** if there is no stable matching ν with $|\mu_S| < |\nu_S|$. A mechanism is *constrained maximal* if it always produces a constrained maximal matching.

Let us revisit the example in the proof of Proposition 2 that shows the lack of maximality of MSD. There, matching ν is stable and assigns more agents than the MSD outcome, revaling that MSD is not even constrained maximal.

Proposition 4. *In the restricted case, MSD is not constrained maximal.*

3.1.2 Adaptive Serial Dictatorship

Given the serious handicaps of MSD, we introduce a new mechanism, which will be shown to be superior to MSD in many aspects. We call this mechanism *Adaptive Serial Dictatorship* and denote it by ASD. We provide the formal definition below.

Adaptive Serial Dictatorship

Step 1. We first tentatively allocate seats to agents. To this end, by following the agent-ordering, we apply the following steps one by one for each agent. For $k \in \{1, .., n\}$,

SubStep $1.k$. Let us consider agent i_k. If there is an available seat such that the adjacent seat is taken by an agent of the same gender as i_k while the other adjacent seat (if any) is not taken by an agent of a different gender, then let i_k receive the best one (with respect to $\rhd_{i_k}$) among such seats. Otherwise, if there is an empty row, then let i_k be seated at his/her favorite seat among those in the empty rows. If neither of these holds, the rows contain three seats, and there is an empty seat whose adjacent is not taken, then let i_k receive the best one among such seats.[19] If none of these hold, then let i_k be unseated.

This procedure terminates by the end of Substep $1.n$. By its termination, if $|\tau_s| = 3$ and there are two rows where one contains only a single man and the other contains a single woman, then we displace the one with a lower priority from her/his seat. We place her/him at the empty non-middle seat in the other agent's row.[20]

Let μ^0 be the tentative matching attained at the end of Step 1. We exclude all the agents in $N \setminus \mu^0_S$ from the problem and they become permanently unassigned. Let us displace the rest from their assignments under μ^0, and each seat becomes available.

Let c be the total number of empty rows under μ^0. If $c > 0$, then we go to Step 2.[21] Otherwise, we go to Step 3.

Step 2. We only consider the top c agents in the ordering. We first let i_1 pick his/her favorite available seat. We remove the agent and the selected row. Among the remaining seats, we repeat the same procedure one by one following the agent-ordering, and it ends after the selection of i_c. We then go to Step 1 in the reduced problem.

Step 3. We have the following exhaustive cases.

Case 1: ***All seats are taken.*** We go to Step 4.

Case 2: ***There is at least one row where only one seat is taken.*** There can be at most two rows where only one seat is taken. We have the following subcases.

[19] There can already be at most one such seat. However, for the sake of coherence, we let him choose his favorite one.

[20] Note that since each agent chooses the best seat in completely empty rows whenever s/he is seated at an empty row, the middle-seat in the row is empty. Therefore, the man-woman pair is not seated next to next.

[21] If $c > 0$, then $N = \mu^0_S$.

Subcase 2.1: ***There are two rows where only one seat is taken.*** This case is possible only if $|\tau_s| = 2$. By Step 1's definition, one of these rows contains a woman, and the other one contains a man. Let us suppose that the top agent under the ordering is a man. The other case follows from symmetric argument. We let the top agent, who is a man, choose his best seat and remove him along with his row from the problem. We apply Step 4 until the top woman takes her turn. She selects the best remaining seat in a completely empty row. We remove her along with the selected seat and its row from the problem. We then go to Step 4.

Subcase 2.2: ***Only one row containing one seated agent whose gender is the same as the top agent.*** We let the top agent choose her/his best remaining seat and remove the selected seat and its row from the problem. We then go back to Step 1.

Subcase 2.3: ***Only one row containing one seated agent whose gender is different from the top agent.*** Suppose $|\tau_s| = 2$. Let agent j be the top agent of the same gender as the agent whose row has an empty seat. We apply Step 4 till agent j. Whenever it is agent j's turn, we let him/her choose his/her best seat in a completely empty row. We remove the row and go to Step 4.

Suppose $|\tau_s| = 3$. We then calculate the total number of empty seats. If it is equal to 3 (this means that there is another row occupied by two agents of the same gender as the top agent), then let the top ranked agent choose his/her best seat. We remove him/her with the selected seat and its row. We then go back to Step 1.[22]

If it is equal to 2, then we consider the top ranked agent and the two other highest ranked agents of the same gender as the former (the top agent). We also consider the top ranked agent of the other gender. Let us write A for the set of these four agents.

The top agent in A chooses his/her favorite seat, and we remove the seat. The next agent in A selects his/her remaining favorite seat. If these two agents are seated in the same row, then we remove the row. We then apply Step 4 until the third ranked agent in A. Whenever

[22] Notice that, throughout the algorithm, whenever we go back to Step 1, the number of removed agent is always less than the number of removed seats. Hence, we can repeat this procedure finitely many times.

it is her/his turn, s/he chooses her/his best seat in a completely empty row, and the other agent in A is seated at her/his best remaining seat in the selected row. We then remove them along with their selected row and go to Step 4. Otherwise—that is, if the second ranked agent in A is not seated in the same row as the top agent in A—then we let the third agent in A choose his/her favorite remaining seat among the empty seats in the rows taken by the first two agents in A. The last agent in A then chooses his/her favorite remaining seat among these rows. We remove these agents along with their rows and go to Step 4.

Case 3: $|\tau_s| = 3$ *and there is a row with two seated agents of different genders.* By Step 1, there can be at most one such row. In this case, the top agent selects his/her favorite seat. The top agent of the other gender selects the best remaining seat in the selected row by the former. We remove these agents, as well as their selected row, from the problem and go to Step 4.

Case 4: $|\tau_s| = 3$ *and none of the above cases hold.* Let d be the total number of empty seats. Note that by our construction $d \in \{1, 2\}$. We consider the following subcases.

Subcase 4.1: $d = 1$. The row containing the empty seat only includes agents with the same gender (the other case is already addressed in Case 3). Without loss of generality, suppose they are both male. Let j_1 and j_2 be the first and second ranked men. Until j_1's turn, we apply Step 4. Whenever it is his turn, we let j_1 choose his best seat in a completely empty row. We also let j_2 choose his best remaining seat in the row where j_1 is seated. We then remove them along with their row. We then go to Step 4.

Subcase 4.2: $d = 2$. There is one row containing two men and one row containing two women. Let us consider the top two agents from the men and women sides and call this set A. Let the top agent in A select his/her favorite seat and remove it. The next agent in A selects his/her favorite remaining seat. If these two agents are seated in the same row, then we remove the row. We then apply Step 4 until the third-ranked agent in A. Whenever it is her/his turn, we let her/him select the best seat in a completely empty row. Then, the remaining agent in A is seated at her/his best remaining seat in the selected row by the former. We then remove them along with their row and go to Step 4. Otherwise, that is, if the second ranked agent in A is not seated in the same row with the top agent in A, then the third agent in A chooses his/her favorite remaining seat among the empty seats in the rows taken by the first two agents in A. The last agent in A then chooses his/her favorite remaining seat among these rows. We remove these rows and go to Step 4.

Step 4. In the reduced problem, one by one following the agent-ordering, we do the following. We start with the first agent (with respect to the agent-ordering) and let him/her choose his/her best available seat. Let us then consider the second agent. Suppose she is a woman (the other case follows from a symmetric argument). Let us calculate the total number of the empty seats in the rows where a woman has already been seated. If this number is equal to the number of unseated women, then we let her choose the best seat in a row where a woman has already been seated. Otherwise,[23] she chooses her best seat in the rows where no man has already been seated. We continue in the same manner until the last agent.

The algorithm terminates by the end of Step 4. The assignments obtained by the above steps define the outcome of the algorithm. We run *ASD* in a problem below.

Example 2. *We consider the same problem given in Example 1. ASD selects its outcome through the following steps.*

***SubStep* 1.1.** *Since all seats are empty, m_1 receives s_1, which is the best ranked seat under $\rhd_{m_1}$.*

***SubStep* 1.2.** *Since s_2 is the only seat whose adjacent seat has been already taken by a man, m_2 receives s_2.*

***SubStep* 1.3.** *Since there is no seat with an adjacent seat is taken by a man, m_3 receives s_3, which is the best ranked available seat under $\rhd_{m_3}$.*

***SubStep* 1.4.** *Since s_4 is the only empty seat whose adjacent seat has been already taken by a man, m_4 receives s_4.*

[23]This number cannot exceed the number of unseated women.

SubStep 1.5**.** *Since there is no seat whose adjacent seat is taken by a woman,* f_1 *receives* s_5*, which is the best ranked available seat under* $\rhd_{f_1}$*.*

Let μ^0 *be the tentative matching attained at the end of Step 1. There is no empty row under* μ^0*. Hence* $c = 0$*, and we continue with Step 3.*

Step 3**.** *Since there is only one row containing one seated agent whose gender is different from the top agent, we consider Subcase 2.3 and apply Step 4 as follows.*

Agents m_1*,* m_2*,* m_3*, and* m_4 *choose* s_1*,* s_3*,* s_2*, and* s_4*, respectively. Agent* f_1 *chooses* s_5 *and the algorithm terminates. Hence, ASD selects matching* ν *where* $\nu_{m_1} = s_1$*,* $\nu_{m_2} = s_3$*,* $\nu_{m_3} = s_2$*,* $\nu_{m_4} = s_4$*, and* $\nu_{f_1} = s_5$*.*

We are now ready to study the properies of ASD. Theorem 1 shows that ASD satisfies stability, efficiency, and strategy-proofness.

Theorem 1. *In the restricted case, ASD is stable, efficient, and strategy-proof.*

One can wonder whether there exists another stable mechanism. Theorem 2 shows that in any problem there exists a unique stable matching, and therefore ASD is the unique stable mechanism.

Theorem 2. *In the restricted case, ASD is the unique stable mechanism.*

Proposition 3 and Theorem 2 imply that ASD is not maximal, yet it is constrained maximal.

Corollary 1. *In the restricted case, ASD is not maximal. However, it is constrained maximal.*

Fortunately, stability does not cause too many empty seats that otherwise would be filled in that, in any problem, a maximal matching can assign at most one more seat than the ASD's outcome.

Proposition 5. *In the restricted case, let* μ' *be a maximal matching at a problem* $\rhd$*. Then,* $|\mu'_S| \leq |ASD_S(\rhd)| + 1$*.*

An interesting question is whether agents would prefer to come earlier in the ordering. In the seat allocation application, this depends on whether buying a ticket earlier is beneficial, and hence incentivizes agents for the earlier purchase. To address this question, we start including $\succ$ in the problem notation and write $(\rhd, \succ)$ instead of $\rhd$.

We say that $\succ'$ is an **improvement** over $\succ$ for agent i if for each $j, k \in N \setminus \{i\}$, $i \succ j$ implies $i \succ' j$, and $j \succ' k$ if and only if $j \succ k$. Mechanism ψ **respects improvements** if there is no problem $(\rhd, \succ)$ and $\succ'$ such that $\succ'$ is an improvement over $\succ$ for agent i, and $\psi(\rhd, \succ) \; P_i \; \psi(\rhd, \succ')$.

Theorem 3. *In the restricted case, ASD respects improvements.*

3.2 The Unrestricted Case

In this section, we assume that there is no gender-based restriction over matchings in the sense that an agent can be seated next to any agent independent of their genders. We start our analysis by first adapting the *Myopic Serial Dictatorship* to the unrestricted case.

3.2.1 Myopic Serial Dictatorship (MSD)

When there is no gender-based restriction, we can apply the MSD mechanism defined in Section 3.1.1 by considering that all agents are of the same gender. For the sake of completeness, we define the MSD mechanism in the unrestricted case.

Myopic Serial Dictatorship in the Unrestricted Case

By following the agent-ordering, the MSD mechanism selects its outcome through following steps. For $k \in \{1, \ldots, n\}$

Step k**.** Let us consider the set of rows with the highest number of empty seats. Among the seats in these rows, agent i_k selects the best ranked one (with respect to $\rhd_{i_k}$). If such a seat does not exist, then i_k stays unassigned.

MSD terminates by the end of Step n. We show below that in the unrestricted case, while MSD becomes maximal, its other negative properties continue holding.

Proposition 6. *In the unrestricted case, MSD is maximal. However, it is not stable, efficient, or strategy-proof.*

Given that MSD continues displaying most of the negative properties, we straightforwardly adapt ASD to the unrestricted case.

3.2.2 Adaptive Serial Dictatorship (ASD)

As explained for MSD, we can apply the ASD mechanism defined in Section 3.1.2 by positing that all the agents are of the same gender. For the sake of completeness, we define ASD mechanism in the unrestricted case.

Adaptive Serial Dictatorship in the Unrestricted Case

Step 1. We first tentatively allocate seats among agents. To this end, by following the agent-ordering, we apply the following steps one by one for each agent. For $k \in \{1, .., n\}$,

SubStep 1.k. Let us consider agent i_k. If there is an available seat whose adjacent seat has been already taken, then let i_k receive his/her favorite seat among such seats. Otherwise, if there is an empty row, then let i_k be seated at his/her favorite seat among the ones in the empty rows. If none of these hold, then let i_k be unseated.

This procedure terminates by the end of Substep 1.n. Let μ^0 be the matching at the end of Step 1. We exclude all the agents in $N \setminus \mu^0_S$ from the problem and let them be permanently unassigned. Let us displace the rest of the agents from their assignments under μ^0, and each seat becomes available to be assigned.

Let c be the total number of empty rows under μ^0. If $c > 0$, then we go to Step 2.[24] Otherwise, we go to Step 3.

Step 2. We only consider the top c agents in the ordering. We first let i_1 pick his/her favorite seat. We remove the agent and the selected row. Among the remaining seats, we repeat the same procedure one by one following the agent-ordering, and it ends after the selection of i_c. We then go to Step 1 in the reduced problem.

[24]If $c > 0$, then $N = \mu^0_S$.

Step 3. We have the following exhaustive cases.

Case 1: ***All seats are taken.*** We go to Step 4.

Case 2: ***There is a row with only one seat is taken.*** No other row contains an empty seat. We let the top agent choose her/his best remaining seat. We remove the agent along with the selected row and go to Step 4.

Case 3: $|\tau_s| = 3$ ***and none of the above cases hold.*** There exists only one row with two seated agents while all the others are fully taken. Let j_1 and j_2 be the first and second ranked agents in the reduced problem. We let j_1 select the best remaining seat. Agent j_2 selects the best remaining seat in the row selected by j_1. We then remove them along with the selected row and go to Step 4.

Step 4. In the reduced problem, we let each agent choose his/her best remaining seat one by one following the agent-ordering.

The algorithm terminates by the end of Step 4. The assignments obtained in the course of the above steps define the outcome of the algorithm.

Note that ASD above is equivalent to the restricted case's ASD whenever all the agents are of the same gender. Therefore, all the earlier positive properties of ASD carry over to the unrestricted case. Moreover, an agent only fails to receive a seat only when all seats are already taken, implying that ASD becomes maximal in the unrestricted case as well.

Proposition 7. *In the unrestricted case, ASD is maximal, stable, efficient, strategy-proof, and respects improvements. Moreover, it is the unique stable mechanism.*

4 Simulations

In this section, we use computer simulations to measure possible gains from replacing the current procedure with ASD. In particular, we calculate the fractions of agents assigned to a seat under the current procedure and ASD under various scenarios based on the number of agents, correlation in preferences over seats, and gender distributions. Moreover, we also calculate the number of instances in which priorities are violated when there is no restriction on the seating. Here, we take MSD as a proxy for the current procedure.

We run separate simulations for 2-seat and 3-seat cases. Under both cases, $|N|$ agents[25] are ordered according to their arrival time and no two agents arrive at the same time. Hence, we have a strict priority order over the agents. Instead of randomly choosing the gender of each agent with the same probabilities, we consider different distributions in which the first half of the agents and the second half of the agents have the same probability of being female and male, respectively. For the first half of the agents, an agent is a female (male) with probability $\delta \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$ $(1-\delta)$, and for the remaining half of the agents, an agent is female (male) with probability $1-\delta$ (δ). Hence, we aim to have a population with equal shares of female and male.

To construct the preference of each agent i we calculate her/his utility from being assigned to each seat s as follows:

$$U_{i,s} = \alpha \times C_s + (1-\alpha) \times D_{i,s},$$

where $C_s \in (0,1)$ represents the common utility received by all individuals from seat s and $D_{i,s} \in (0,1)$ represents the individual specific utility received by agent i from seat s. Both C_s and $D_{i,s}$ are selected from i.i.d. standard uniform distribution. The correlation between the preferences of the agents is captured by variable $\alpha \in \{0, 0.25, 0.5, 0.75, 1\}$. The higher the α, the more correlated preferences are. The calculated utility values of agents over the seats are used to construct the ordinal preferences of agents over the seats.

For 2-seat case, we set the number of rows to 50 and the number of seats to 100. We consider five different cases based on the number of agents, namely 80, 90, 100, 110, and 120. When the number of agents is less than the number of seats, there is less competition for the seats and, in the unrestricted case, every agent can be seated. On the other hand, when the number of agents is more than the number of seats, the competition is more fierce, and some

[25] We take $|N|$ as an even number.

agent has to be unassigned. Our theoretical results imply that when the number of agents is 80 or 90, ASD assigns each agent to a seat. Moreover, for the remaining cases, ASD will waste at most one seat. Our simulations verify these theoretical results. On the other hand, under MSD, we observe many wasted seats, specifically for a lower level of δ. The MSD performs poorly when the number of agents is less than the number of seats. This follows from two key facts: (1) when there are fewer agents than seats, there is less competition for the seats and agents are not seated due to the skewed distribution of arrivals, and (2) when there are more agents than seats, we can find enough agents to fill them. We also observe that α does not affect the ratio of seated agents under MSD to that under ASD. We present our simulation results for the 2-seat case for $\alpha = 0.5$ in Figure 2.

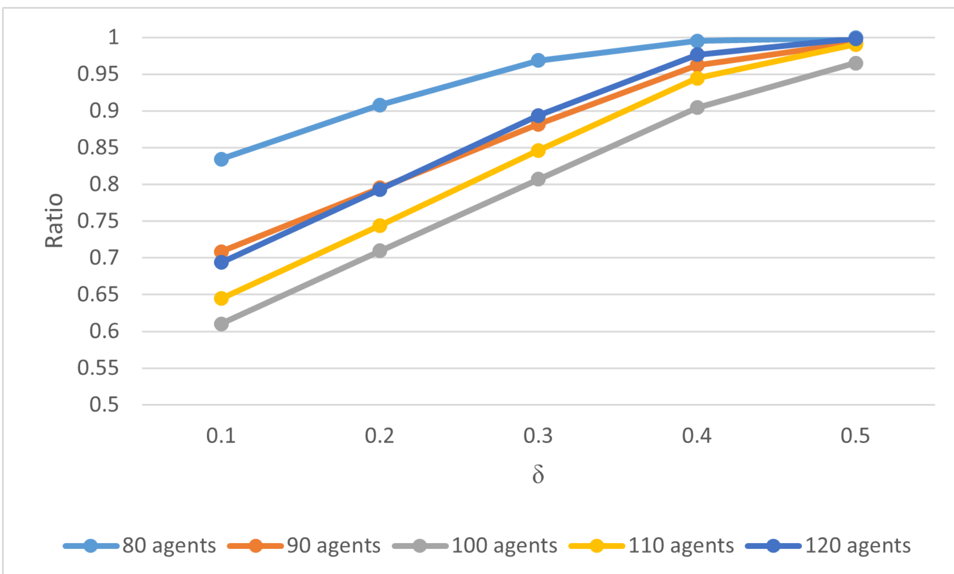

Figure 2: Ratio of seated passengers under MSD and ASD (2-seat case)

For the 3-seat case, we set the number of rows as 40 and the number of seats as 120. We consider five different cases based on the number of agents, namely 100, 110, 120, 130, and 140. Our theoretical results imply that for the cases of 100 and 110 agents independent of the gender distribution, ASD can seat all agents. Moreover, for the remaining cases, under ASD, there will be at most one unfilled seat. Our simulations verify these theoretical results. On the other hand, under MSD, we observe many wasted seats, specifically for the lower level of δ independent of the number of agents. We also observe that α does not affect the ratio of seated agents under MSD to the one under ASD. We represent our simulation results for the 3-seat case for $\alpha = 0.5$ in Figure 3.

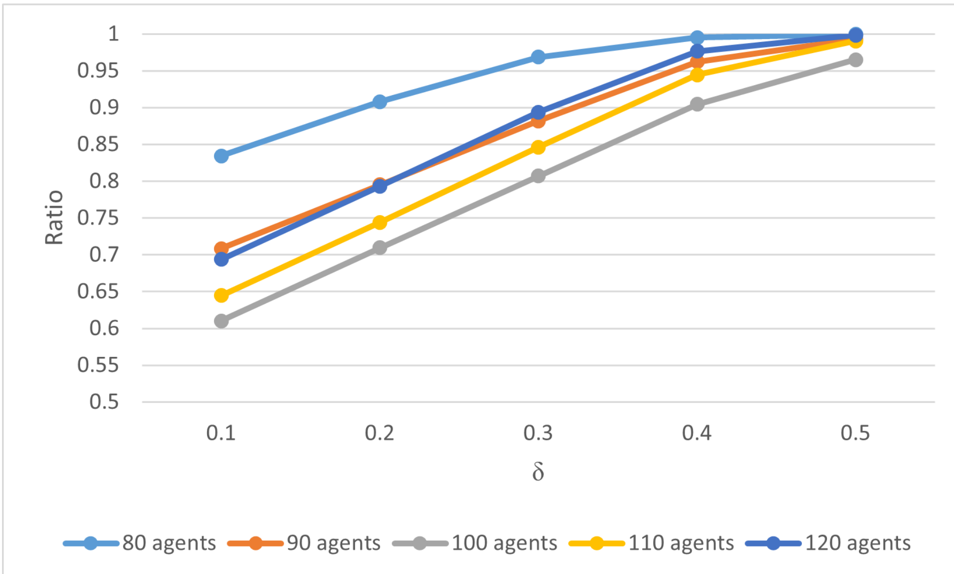

Figure 3: Ratio of seated passengers under MSD and ASD (3-seat case)

We also conduct a simulation analysis under the unrestricted case, i.e., females and males can sit next to each other. Under the unrestricted case, both MSD and ASD do not waste any seats. That is, either all agents are seated or all seats are allocated to some agents. In this unrestricted case, stability implies that no agent would like to swap his/her assignment with another agent with lower priority. As shown in Section 3.2, ASD is stable. By using our simulations, we calculate the fraction of agents who would like to swap their assignment with a lower priority agents under MSD. The results under 2-seat and 3-seat cases for different levels of α and number of agents are given in Figures 4 and 5, respectively. As the preferences become more correlated, we observe higher level of priority violations in all cases. Moreover, we observe higher priority violations when there are less agents than seats. This is due to the fact that, when preferences are correlated, agents with lower priority are seated in the unfilled rows since they pick later than the higher priority agents.

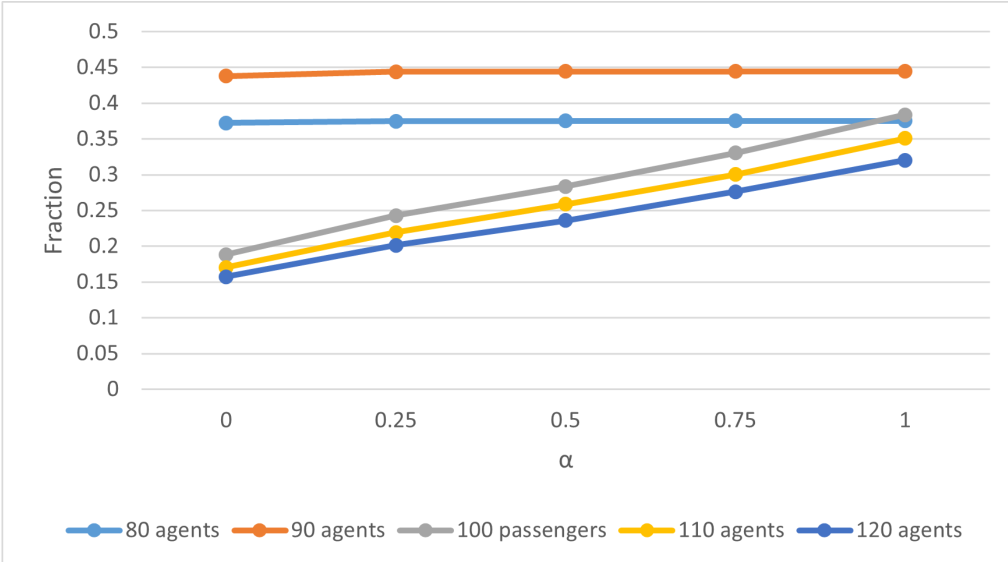

Figure 4: Fraction of agents whose priorities violated under MSD (2-seat case)

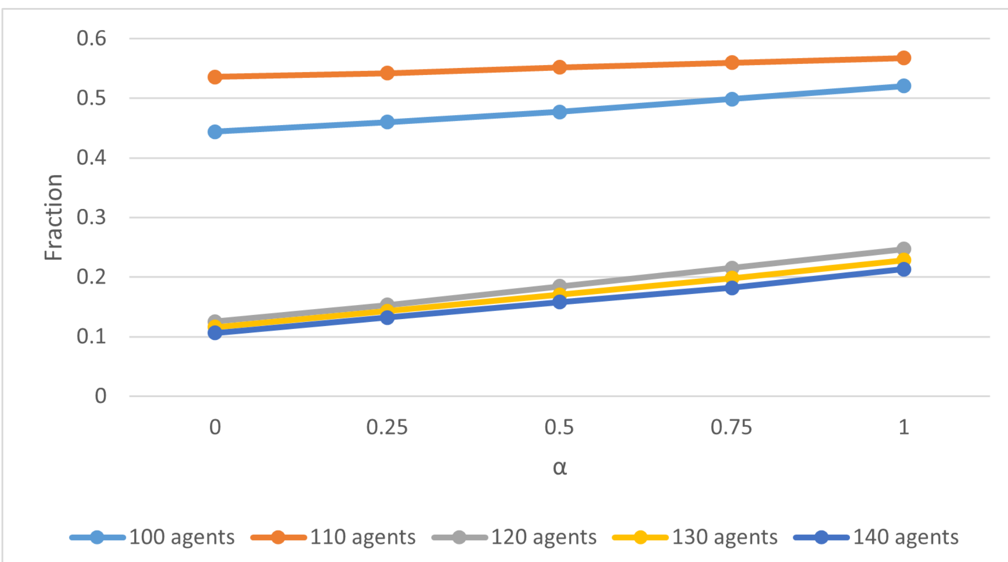

Figure 5: Fraction of agents whose priorities violated under MSD (3-seat case)

5 Discussion

A natural extension of our framework is to incorporate couples. Even under gender-based seating restrictions, one may expect that couples can be seated in adjacent seats. Indeed, this is the practice for the Turkish Railways. Fortunately, our algorithm can easily be modified to handle the inclusion of couples in the problem. In the first step, as described above, we treat couples as different agents coming back to back in the ordering (it does not matter who comes first). We elicit one seat ordering from couples. We then apply the same Step 1 of the algorithm with the exception that a couple is not seated whenever either the man or the woman in the couple does not receive a seat. Next, they move to the second stage. The second stage works the same with the exception that whenever the woman (the man) in a couple selects a seat, she (he) selects their favorite seat with an empty seat next to it. Otherwise, he/she is assigned as before. In the second stage, if an agent cannot be seated, it is because of the seat allocation of a couple, i.e., the middle seat is given to an agent of a different gender from the former. In this case, we can swap the couple's seats, and the agent can be seated.

Another issue we want to elaborate on is the practical implementation of ASD. We statically define the mechanism in the sense that all the agents are pooled, and then it calculates the outcome. However, this is not practical, as agents make reservations over time, and they need to know whether they can reserve a seat right away. Fortunately, ASD can be implemented dynamically to address this concern. Its first step can be run for each new agent's arrival, and the agent can be informed whether she will be seated. Notice that whether or not an agent is seated only depends on the agents who have arrived earlier. This procedure can be conducted subject to a deadline, depending on the trip. Then, Step 2 determines the final seat assignments. The agents can thus be informed about their seats before the trip. In fact, this implementation is very similar to the seat allocation procedure followed by airlines. Many airline companies sell tickets to the passengers without assigning a specific seat at the time of purchase. Passengers are usually informed about their seats when they check-in 24 hours before departure.

Lastly, we want to discuss to what extent our analysis extends to the weak preferences domain. This is worthy to touch on as agents' seat rankings may very well not be strict, implying that their preferences over matchings may involve indifferences. We can adapt ASD to this case as follows. We can first obtain a strict ranking over seats by applying a tie-breaking rule. Then, with the obtained strict ordering, we can invoke ASD to find a matching. This matching, however, may not be efficient. To fix this, we can utilize efficiency-improving cycles, which have been well-studied in the literature (for instance, see Erdil and Ergin (2008) and Kesten and Ünver (2015)). Cycle construction would take a simple form because the number of seated agents' in an agent's row at the ASD matching needs to be fixed. Therefore, we only need to consider agents' seat rankings in constructing cycles. We can define stability preserving and efficiency-improving cycles similar to those that have already been defined and used in the literature. These cycles will also need to take care of feasibility constraints. Such a mechanism would be feasible, efficient, stable, and constrained maximal. However, because of indifferences, there will be multiple stable matchings, implying that the characterization result (Theorem 2) will no longer hold. Moreover, the mechanism would be manipulable. It may be a fruitful research direction to study the weak preference domain further.

6 Conclusion

Seating restrictions on public transportation are exercised in many countries. In most countries, a certain number of seats is reserved within a section of the train (or bus) for women. Different from such reservations, in some countries, such as Turkey, women and men are not allowed to sit in adjacent seats. When passengers choose their seats, this restriction may result in various problems, including wasted seats and stability violations. In this paper, we introduce a mechanism that can be easily and practically implemented. This mechanism has appealing theoretical properties, including stability, efficiency, and strategy-proofness. Apart from public transportation, it can be applied to other allocation problems where two different types of agents cannot be allocated to two specific objects at the same time. One such problem is hospital bed allocation during a pandemic where a patient with a contagious illness cannot be assigned to the same room with other patients without a contagious illness.

References

Abdulkadiroğlu, A., P. A. Pathak, and A. E. Roth (2009): "Strategy-proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match," *American Economic Review*, 99, 1954–1978.

Abdulkadiroğlu, A. and T. Sönmez (2003): "School Choice: A Mechanism Design Approach," *American Economic Review*, 93, 729–747.

Balinski, M. and T. Sönmez (1999): "A tale of two mechanisms: student placement," *Journal of Economic Theory*, 84, 73–94.

Erdil, A. and H. Ergin (2008): "What's the Matter with Tie-breaking? Improving Efficiency in School Choice," *American Economic Review*, 98, 669–689.

Ergin, H., T. Sönmez, and M. U. Ünver (2017): "Dual-donor organ exchange," *Econometrica*, forthcoming.

Fisher, J. C. and I. E. Hafalir (2016): "Matching with aggregate externalities," *Mathematical Social Sciences*, 81, 1–7.

Freisleben, B. and G. Gleichmann (1993): "Controlling airline seat allocations with neural networks," in *[1993] Proceedings of the Twenty-sixth Hawaii International Conference on System Sciences*, vol. 4, 635–642.

Hafalir, I. (2008): "Stability of marriage with externalities," *International Journal of Game Theory*, 37, 353–369.

Kesten, O. (2010): "School Choice with Consent," *Quarterly Journal of Economics*, 125, 1297–1348, *Quarterly Journal of Economics*, forthcoming.

Kesten, O. and M. U. Ünver (2015): "A Theory of School Choice Lotteries," *Theoretical Economics*, 10, 543–595.

Pycia, M. and M. B. Yenmez (2019): "Matching with externalities," .

Roth, A. E., T. Sönmez, and M. U. Ünver (2004): "Kidney Exchange," *Quarterly Journal of Economics*, 119, 457–488.

Sasaki, H. and M. Toda (1996): "Two-sided matching problems with externalities," *Journal of Economic Theory*, 70, 93–108.

Sawaki, K. (1989): "AN ANALYSIS OF AIRLINE SEAT ALLOCATION," *Journal of the Operations Research Society of Japan*, 32, 411–419.

Sönmez, T. and T. Switzer (2012): "Matching with (branch-of-choice) contracts at United States Military Academy," Forthcoming, *Econometrica*.

Trapp, A. C., A. Teytelboym, A. Martinello, T. Andersson, and N. Ahani (2020): "Placement Optimization in Refugee Resettlement," *Operations Research*, forthcoming.

Yuan, W. and L. Nie (2020): "Optimization of seat allocation with fixed prices: An application of railway revenue management in China," *PLoS ONE*, 15.

A Proofs

Proof of Proposition 1. Let μ be a stable matching. Assume for a contradiction that it is not efficient. Assume for a contradiction that μ is not efficient. Let μ' be a matching such that for each $i \in N$, $\mu' \; R_i \; \mu$, where this relation strictly holds for some agent j. This, as well as the stability of μ, implies that $\mu_S = \mu'_S$.

Let $W = \{i_k \in N : \; \mu' \; P_{i_k} \; \mu\}$. By supposition, $W \neq \emptyset$. Let i_k be the last agent in W (with respect to the agent-ordering). Note that as $\mu_S = \mu'_S$, $\mu_{i_k} \neq \emptyset$. For each $k' < k$, $\mu' \; R_{i_{k'}} \; \mu$, $\mu' \; P_{i_k} \; \mu$, and $\mu'_S = \mu_S$. This, however, contradicts the stability of μ, which finishes the proof. □

Proof of Proposition 2. **MSD is neither stable nor efficient** Let $N = N^m = \{m_1, m_2, m_3, m_4\}$, $S = \{s_1, s_2, s_3, s_4\}$, $\tau_{s_1} = s_2$, $\tau_{s_3} = s_4$. We will be using the same seats with the same row pattern throughout the proof. Let $m_1 \succ m_2 \succ m_3 \succ m_4$. Agents' strict rankings over S are:

$$s_1 \rhd_{m_1} s_2 \rhd_{m_1} s_3 \rhd_{m_1} s_4$$

$$s_2 \rhd_{m_2} s_1 \rhd_{m_2} s_3 \rhd_{m_2} s_4$$

$$s_3 \rhd_{m_3} s_1 \rhd_{m_3} s_2 \rhd_{m_3} s_4$$

$$s_2 \rhd_{m_4} s_1 \rhd_{m_4} s_3 \rhd_{m_4} s_4$$

In this problem, MSD selects matching μ such that $\mu_{m_1} = s_1$, $\mu_{m_2} = s_3$, $\mu_{m_3} = s_2$, and $\mu_{m_4} = s_4$. Matching μ is not stable, as there exists another matching ν such that $\nu_{m_1} = s_1$, $\nu_{m_2} = s_2$, $\nu_{m_3} = s_3$, and $\nu_{m_4} = s_4$, rendering the violation of stability under μ. Moreover, ν Pareto dominates μ. Therefore, MSD is neither stable nor efficient.

MSD is not maximal: Let $N^m = \{m_1, m_2\}$, $N^f = \{f_1\}$, with $m_1 \succ m_2 \succ f_1$. Agents' strict rankings over S are:

$$s_1 \rhd_{m_1} s_2 \rhd_{m_1} s_3 \rhd_{m_1} s_4$$

$$s_2 \rhd_{m_2} s_1 \rhd_{m_2} s_3 \rhd_{m_2} s_4$$

$$s_3 \rhd_{f_1} s_1 \rhd_{f_1} s_2 \rhd_{f_1} s_4$$

In this problem, MSD selects matching μ where $\mu_{m_1} = s_1$, $\mu_{m_2} = s_3$, and $\mu_{f_1} = \emptyset$. Matching μ is not maximal because there exists another matching ν such that $\nu_{m_1} = s_1$, $\nu_{m_2} = s_2$, and $\nu_{f_1} = s_3$. Hence, MSD is not maximal.

MSD is not strategy-proof: Let $N = N^m = \{m_1, m_2, m_3\}$, with $m_1 \succ m_2 \succ m_3$. Agents' strict rankings over S are:

$$s_1 \rhd_{m_1} s_2 \rhd_{m_1} s_3 \rhd_{m_1} s_4$$

$$s_2 \rhd_{m_2} s_1 \rhd_{m_2} s_3 \rhd_{m_2} s_4$$

$$s_1 \rhd_{m_3} s_2 \rhd_{m_3} s_3 \rhd_{m_3} s_4$$

In this problem, MSD selects matching μ where $\mu_{m_1} = s_1$, $\mu_{m_2} = s_3$, and $\mu_{m_3} = s_2$. However, if agent m_1 reports $s_3 \rhd'_{m_1} s_4 \rhd'_{m_1} s_1 \rhd'_{m_1} s_2$, then MSD selects matching ν such that $\nu_{m_1} = s_3$, $\nu_{m_2} = s_2$, and $\nu_{m_3} = s_1$. Note that $\nu_{m_1} \; P_{m_1} \; \mu_{\mu_1}$, showing that agent m_1 profitably manipulates MSD. □

Proof of Proposition 3. Consider a problem with $N^m = \{m_1, m_2, m_3, m_4\}$, $N^f = \{f_1\}$, and $S = \{s_1, s_2, s_3, s_4\}$. Let $\tau_{s_1} = s_2$ and $\tau_{s_3} = s_4$. Let $m_1 \succ m_2 \succ f_1 \succ m_3 \succ m_4$.

Independent of the agent preferences, m_1, m_2, and f_1 have to be seated in any stable matching. However, this implies three seats in total are allocated at any stable matching. On the other hand, we can have all four agents in N^m seated at a matching, showing the incompatibility between stability and maximality. □

The following two lemmas will be critical to the rest of the proofs.

Lemma 1. *Let $\rhd$ be a problem. Once ASD reaches Step 4, in the associated reduced problem, all the agents are seated, and no seat is left empty. In the restricted case, no pair of agents of different gender is seated in the same row in Step 4. Moreover, in both restricted and unrestricted cases, in the associated reduced problem, no agent receives a better outcome without hurting someone else with a higher priority while continue assigning all the agents.*

Proof. By construction, once ASD reaches Step 4, in the reduced problem, there are just enough empty seats to let the remaining agents receive a seat in a way that no pair of agents of different gender is to be seated in the same row. One by one following the agent-ordering, a woman (man) chooses her (his) favorite empty seat in rows where another woman (man) has been already seated if the total number of empty seats in the rows where a woman (man) has been already seated is just equal to the total number of unassigned women (men). Otherwise, s/he chooses the favorite seat in a row where no man (woman) has been already seated. This construction ensures that no woman (man) selects a seat in a completely empty row that would prevent some man (in the reduced problem) from being unseated due to the gender restriction. As agents choose their favorite empty seats one by one subject to the row selection condition above, no agent can obtain a better outcome without hurting someone else with a higher priority while giving a seat to each agent. □

Lemma 2. *Let $ASD(\rhd) = \mu$. Let μ' be another matching where $\mu'_S = \mu_S$. If, for some i_k, $\mu' \; P_{i_k} \; \mu$, then there exists some agent $j \in U(i_k)$ such that $\mu \; P_j \; \mu'$.*

Proof. Let ψ denote ASD. Let μ'' be the outcome of Step 1 of ψ. Since $\mu'_S = \mu_S$, we have $\mu'_S = \mu''_S$. We next claim that the number of empty rows[26] under μ' cannot exceed that under μ''. Let i_k be an agent such that s/he is seated at a new row in Step 1 of ψ. Without loss of generality, let us assume that he is a man. Note that whenever it is agent i_k's turn, there cannot be a row containing two seated agents of different gender. This is because, by definition, the agent arriving later among these two is to be seated in a completely empty

[26] A row is empty if no seat is taken in this row.

row (we know that there exists such a row as i_k is seated at a completely empty row). Therefore, the only reason why agent i_k is seated at a new row is that, by his turn, there is no row containing a man and an empty-seat. If $|\tau_s| = 2$, then it directly implies that no row is wasted in ψ, which supports the claim. Let us now assume that $|\tau_s| = 3$. If all the rows including the taken seats by the earlier agents are full, or any row containing an empty seat includes two women, then the only way to give a seat to agent i_k is to open a new row, implying the claim. Otherwise, there can be a row containing only one woman (note that by our preferential supposition, the woman takes a non-middle seat in the row), say j. However, in Substep 1.k of ψ, instead of placing agent i_k at woman j's row, he is placed at a completely empty row. If there is no other agent after i_k (that is, i_k is the last agent in N), then agent i_k is displaced from his seat and seated at the available non-middle seat at the j's row. Therefore, no row is wasted. Otherwise, if there exists an agent after i_k, then a new row would have to be opened for him/her whenever i_k was originally placed at woman j's row. However, as in ψ, whenever i_k is placed at a new row, the later coming agent is placed either at woman j's row man i_k's row. Therefore, no row is wasted in ψ. This proves the claim.

Let c be the total number of empty rows under μ''. Then, in the course of Step 2, the top c agents choose their favorite seat one by one following the agent-ordering. Moreover, these empty rows are used for the sake of their welfare in that all the other seats in their selected rows are removed from the problem so that each of them is ultimately seated alone. If any of these top c agents receives a different seat under μ', then s/he prefers μ to μ'. Therefore, we suppose all top c agents receive the same seat under μ and μ' and they sit alone in the corresponding row. After Step 2, we redo Step 1 in the reduced problem, and so forth. Because of our claim above, in each Step 1 application, the number of empty rows is maximal given the set of seated agents in the reduced problem. All these show that under μ, no agent who is seated alone can be better off without hurting any agent who comes earlier than himself (herself).

Let us now consider Step 3. The algorithm moves to Step 3 whenever there is no empty row left. Note that all the agents who received their assignments and were removed earlier are seated alone in their rows. If, at the end of Step 1 in the last reduced problem, no seat is left empty (this corresponds to Case 1 of Step 3), then this means that all the agents in the reduced problem are to be seated in full rows. We then go to Step 4. By Lemma 1, each agent in the reduced problem in Step 4 is seated in completely filled rows, and there is no way of making someone among them better off without hurting any one coming before in the ordering while keeping the same set of assigned agents as under μ.

Let us next consider Case 2 in Step 3. At the end of the last Step 1 application, if $|\tau_s| = 3$, then there can be at most one row containing only one agent, say i_k (by Step 1's definition, it is immediately apparent to see that there cannot be two rows containing only one agent). Let us assume that i_k is a woman. The other case directly follows from symmetric arguments. By the definition of Step 1 of ψ, this implies that no row contains a pair of agents of different genders and, moreover, each row containing a woman other than i_k is full.

We have two cases to consider. First, the top agent, say j, in the reduced problem is a woman (the top agent can be i_k). Note that this case corresponds to Subcase 2.2 in ψ. This implies that agent j can be seated alone in a row. Hence, we let her choose her best seat and remove her, as well as her selected row, from the problem. We then invoke Step 1 in the reduced problem. Otherwise, agent j, the top agent in the reduced problem, is a man. Note that this case corresponds to Subcase 2.3 in ψ. We then calculate the total number of empty seats. We already have two empty seats in agent i_k's row. We know that no row containing a woman contains an empty seat. Therefore, if there is an additional empty seat, then it has to come from a row containing two men. By the definition of Step 1, at most one row contains two men, and all the other rows are fully taken. Therefore, the total number of empty seats is either 3 or 2.

Let us assume that it is 3. This means that there is one row containing two men. Recall that we already have a row containing only woman i_k. All the other rows are fully taken. This case implies that the top agent, who is a man, can be seated alone. Therefore, we let him choose his best remaining seat in the reduced problem and remove him, as well as his selected row, from the problem. We then run Step 1 in the reduced problem.

Otherwise, the total number of empty seats is 2. This means that all the rows, except the one containing i_k, are fully taken. In this case, the top agent, who is a man, cannot be seated alone. However, he can be seated in a pair. As the only unfilled row contains only one woman, i_k, and all the other rows are fully taken, it implies that three men and one woman can be seated in pairs, and all the others are to be seated in full rows. Let A be the set with the three top-ranked men and the top-ranked woman. The first top two agents in A choose their favorite available seat one by one, starting with the former. If they are seated in the same row, then we run Step 4 till the top third agent in A, ensuring that all the other agents are to be seated in full rows. Then, the top third agent in A picks his (her) favorite remaining seat, and the remaining agent in A picks his (her) favorite remaining seat in the row selected by the third agent. Otherwise, if the first two agents are seated at different rows, then the third agent picks his (her) favorite seat among the remaining ones in those two rows, and the fourth agent picks his (her) favorite seat in the remaining row. Hence, these fours agents' welfare is maximized as much as possible in the reduced problem. We remove them, along with their selected rows. The algorithm then goes to Step 4. All the rest must be seated in full rows. All these, as well as Lemma 1, show that under μ, no agent can be better off without hurting someone coming earlier than himself (herself) while keeping the same set of assigned agents as under μ.

On the other hand, if $|\tau_s| = 2$, there can be at most two rows, each containing one agent. Let us assume that there are two such rows (this case corresponds to Subcase 2.1 in ψ). By Step 1's definition, one of these rows contains a man, and the other contains a woman. This implies that all the other rows are full, and hence only one man and one woman can be seated alone. Without loss of generality, assume that the top agent is a man. The algorithm lets the top agent choose his best seat and removes him along with his row. Then, Step 4 is applied till the top woman's turn. Whenever it is her turn, she selects the best remaining seat in an empty row. We then remove her along with the selected row. The algorithm then goes to Step 4. All these, as well as Lemma 1, imply the result.

Suppose there is only one row containing only one agent and $|\tau_s| = 2$. This implies that all the other rows are full. Let us assume that the row contains a man. The other case follows from symmetric arguments. As all the other rows are full, it implies that only one man can be seated alone. If the top agent is a man, then he picks his best seat, and we remove his row and go back to Step 1. Otherwise, the top agent is a woman. In this case, only one woman can be seated alone. The algorithm works in the same way as above except whenever it is the top woman's turn, she selects her favorite seat from an empty row, and the selected row is removed from the problem, implying the result.

Let us now consider Case 3 in Step 3. Suppose there is a man-woman pair who is seated in the same row. This case can only happen for $|\tau_s| = 3$. Note that there cannot be more than one such rows. This implies that all the other rows are completely full. Hence, only one man and one woman can be seated in a pair while all the rest have to seated in full rows. The algorithm then selects the top woman and top man. The top agent among these chooses his (her) best seat in the reduced problem, and the other is seated at her (his) favorite seat in the selected row by the former. These two agents along with their row are removed from the problem. As the rest are to be seated in full rows, the algorithm goes to Step 4. This, as well as Lemma 1, implies the result.

Let us consider Case 4. The algorithm reaches this case only when $|\tau_s| = 3$. Suppose there is only one empty seat. This means that one row contains two women (or men), and all the other rows are fully taken (note that the case where a row contains a man and a woman is addressed in Case 3). Suppose that the non-full row contains two women. This case implies that only two women can be seated in a pair, and all the others have to be seated in full rows. In ψ, until the first-ranked woman, we apply Step 4. Whenever it is her turn, she chooses her favorite available seat in a completely empty row. We also let the second-ranked woman choose her best available seat in that row. We then remove them along with their row. We then proceed to Step 4 for the rest. Hence, the two top-ranked women enjoy having an empty seat while all the rest are seated through Step 4. This, as well as Lemma 1, implies the result.

Let us next consider the case where there are two empty seats in total. This implies that there are two rows, each containing one empty seat (note that no row contains a pair of agents of different genders). All the other rows are fully taken. This implies that only two men and two women can be seated in pairs. In ψ, the two top men and two top women are selected. Let us write A for the set of these four agents. The rest of this case is the same as the relevant part in the case above where we consider only one row containing one agent and the total number of empty seats is 2.

By the definition of Step 1, all these cases are exhaustive—that is, there is no other case left. Moreover, in each case where the algorithm goes back to Step 1, some row is removed. Hence, the algorithm ultimately falls into Step 4. Hence, by Lemma 1, we have the result. □

Proof of Theorem 1. Consider an arbitrary problem $\triangleright$. Let $ASD(\triangleright) = \mu$. We first show that μ is stable. Let us pick an agent i_k with $\mu_{i_k} = \emptyset$. Then, by the definition of Step 1 of ASD, it is not feasible to place agent i_k unless some agent in $\mu_S \cap U(i_k)$ loses his seat under μ and is unassigned. Let us now assume that $\mu_{i_k} \neq \emptyset$. By Lemma 2, in order to improve agent i_k's outcome while keeping the same set of assigned agents, some better ranked agent has to be worse off. Therefore, stability is not violated. All these show that μ is stable. By Proposition 1, stability implies Pareto efficiency.

Finally, we show that no agent can benefit from misreporting under ASD. First notice, agents cannot affect the set of seated agents. Moreover, no agent can affect the seat assignment of the earlier agents. All these, as well as Lemma 2, imply that no agent can benefit by misreporting his preferences under ASD, i.e., ASD is strategy-proof. □

Proof of Theorem 2. Consider an arbitrary problem $\rhd$. Let $ASD(\rhd) = \mu$. By Theorem 1, μ is stable. Assume for a contradiction that there exists another stable matching μ'.

We first claim that $\mu_S \setminus \mu'_S = \emptyset$. Assume for a contradiction that $\mu_S \setminus \mu'_S \neq \emptyset$. Let i be the best ranked agent in $\mu_S \setminus \mu'_S$ according to $\succ$. This implies that for each $j \in U(i)$, either $j \in \mu_S \cap \mu'_S$ or $\mu_j = \mu'_j = \emptyset$. Note that $\mu'_i = \emptyset$ and $\mu_i \neq \emptyset$. Therefore, we have a matching μ where $\mu_i \neq \emptyset$ and $U(i) \cap \mu'_S \subseteq U(i) \cap \mu_S$. This contradicts the stability of μ'. Hence, we have $\mu_S \subseteq \mu'_S$. This, as well as the stability of μ, implies that $\mu_S = \mu'_S$.

Let $\mu_S = \{i_1, .., i_n\}$ be the enumeration of the assigned agents on the basis of the agent-ordering. Suppose that $\mu \neq \mu'$. Without loss of generality, let i_k be the first agent who is not indifferent between matchings, and, without loss of generality, let $\mu \ P_{i_k} \ \mu'$. Then, μ' cannot be stable, because of the violation of the stability's second condition, contradicting the stability of μ'. □

Proof of Proposition 5. First, if each agent receives a seat under ASD, then there is nothing to prove, as it already implies the maximality of the ASD outcome. Suppose that agent i does not receive a seat, implying that s/he is not seated in Step 1 of ASD. For the rest of the model, we only consider the Step 1 of ASD.

Let us first assume that $|\tau_s| = 2$. Agent i cannot be seated in Step 1 whenever either all seats are already taken or there is only one seat left whose adjacent seat is taken by an agent, say j, of the other type. If agent j is the last agent in the ordering of his/her gender, then one seat is left empty. Otherwise, it is taken as well, implying that no seat is left unassigned. These imply that ASD's outcome, if not maximal, assigns one seat less than a maximal matching.

Let us now consider $|\tau_s| = 3$. Agent i cannot be seated in Step 1 only when there is no seat left in his/her turn, or there is an empty seat in a row, but its adjacent seat is already taken by an agent of different gender. We now claim that there cannot be more than one empty seat at the Step 1 outcome of ASD. Assume for a contradiction that there are two empty seats. We have two cases. We may have two non-full rows where one of them contains two men, and the other one contains two women (all the other rows are full). In this case, agent i would have received a seat. Otherwise, we may have a row containing two empty seats. In this case, again agent i would have received a seat. This, in turn, shows that at most one seat is left empty under ASD, finishing the proof. □

Proof of Theorem 3. Consider an arbitrary problem $(\triangleright, \succ)$. Let $\succ'$ be an improvement over $\succ$ for agent i. Let $ASD(\triangleright, \succ') = \mu'$ and $ASD(\triangleright, \succ) = \mu$. First, if $\mu_i = \emptyset$, then there is nothing to prove. Let us suppose that $\mu_i \neq \emptyset$. Let i be the k^{th} agent in the ordering under $\succ$, that is, $i = i_k$.

By the definition of Step 1 of ASD, it is immediately apparent that $\mu_S = \mu'_S$. Then, by the stability of μ and μ', it must be that $\mu \ R_{i_1} \ \mu'$ and $\mu' \ R_{i_1} \ \mu$. That is, i_1 is indifferent between μ and μ'. The same is true for all agents until agent i_k under $\succ'$.

Assume for a contradiction that $\mu \ P_{i_k} \ \mu'$. Then, μ' cannot be stable, as $\mu_S = \mu'_S$, all the agents arriving earlier than i_k are indifferent between μ and μ', and agent i_k prefers the former to the latter. This shows that $\mu' \ R_i \ \mu$, finishing the proof. □

Proof of Proposition 6. An agent is unassigned under MSD only when no empty seat is left. This shows that MSD is maximal. In the proof of Proposition 2, we show the lack of stability, efficiency, and strategy-proofness of MSD in problems where all the agents are male. Therefore, the same examples show that MSD is not stable, efficient, or strategy-proof in the unrestricted case as well. □