Seat Allocation Problem in Public Transportation*

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Abstract

We study a seat allocation problem in public transportation. Motivated by different practices in advanced and developing countries, we consider gender-based restrictions which require no pair of passengers of different genders can be seated next to each other. We first show that the commonly used procedure suffers from serious handicaps including welfare losses due to unfilled seats. We then introduce a new mechanism that avoids all these deficiencies while also satisfying some other desirable properties. We also show that our proposal is the only mechanism respecting the rankings over passengers (e.g., booking times) while minimizing unfilled seats. We run simulations to quantify the gain from replacing the current procedure with our proposal. Finally, we show that our analysis can be easily applied to the case in which there are no gender-based restrictions.

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1 Introduction

"In 2015, the United Nations Population Fund commissioned a study on sexual harassment against women on public buses and trains in Sri Lanka. A total of 2,500 women were surveyed for the study and 90% of them said they had experienced sexual harassment."[1]

"According to a survey by Lady Doak College, almost all women respondents faced sexual harassment on public buses in Maduari, a city in India. In fact, 29% of them said that they had experienced it more than five occasions."[2]

Albeit less significantly, sexual harassment in public buses is also an issue in developed countries. According to The European Union Agency for Fundamental Rights, up to 55% of women within the European Union had experienced sexual harassment in public transportation.[3]

Various actions, including harsher penalties and ladies-only buses, have been taken in several countries to at least mitigate the harassment problem. While gender-segregated transportation (GST) seems to be a more definitive solution, it has been criticized from different angles, including its inefficiencies, economic cost, mobility, and the new restrictive social norms it entails. For instance, variation in demand can result in inefficient use of GST. Below is an excerpt from a policy report supported by World Bank:[4]

"A study of pink buses introduced in Lahore, Pakistan, in 2012 found that the need for financial subsidies escalated over time because of the low ridership (Daha 2014). More

[1] `https://srilanka.unfpa.org/sites/default/files/pub-pdf/FINAL%20POLICY%20BRIEF%20-%20ENGLISH_0.pdf`

[2] `https://timesofindia.indiatimes.com/city/madurai/sexual-harassment-high-on-buses/articleshow/90088285.cms`.

[3] `https://fra.europa.eu/sites/default/files/fra_uploads/fra-2014-vaw-survey-main-results-apr14_en.pdf`

[4] `https://www.ifc.org/wps/wcm/connect/c4ce4844-ff6a-4537-b4c4-3ff956d5f3ee/062020+IFC+Gender+Segregated+Ride+Hailing.pdf?MOD=AJPERES&CVID=naU5q5r`.

common are reports of overcrowding. A commuter in the Indian state of Kerala observed that, although nearly the same number of men and women travel in buses during non-peak hours, women are often the ones who end up without a seat."

The same report argues that GST tends to increase women's mobility, enhancing their participation in the labor force and ability to attend school. At the same time, this mode of transportation is often unsustainable as the economic costs of GST can lead to the suspension of these services altogether. GST also entails a new restrictive social norm that forces women to utilize it as, otherwise, harassment is partially deemed as their fault.

A more compromising alternative to GST is gender-based seating (GBS). A male-female pair cannot be seated next to each other unless they make a reservation together. This practice has been in use in some countries, including India, Sri Lanka, Japan, and Turkey.[5,6]

Reservations under GBS are commonly done on a first-come-first-serve basis. That is, by their login time to the booking application, passengers see the available seats that they can book depending on their gender. They then reserve the seat of their choice. For instance, we include two booking screenshots below, one from the Turkish intercity railway, and the other from the intercity bus transportation in India.

Figure 1 is a screenshot of the Turkish Railway booking system. Accordingly, if a customer is male (female) and attempts to reserve a seat whose adjacent has already been taken by a female (male), the system does not let him (her) go through and gives a warning.[7] In Figure 2, exhibiting the case in India, the system directly shows the seats, shown by unfilled red squares, that can only be taken by a woman. Notice that the seats next to these have already been taken by another woman.

An unavoidable implication of the GBS is that some seats might be wasted. However,

[5] `https://globalpressjournal.com/asia/sri_lanka/sri-lankan-entrepreneur-develops-lady-seat-option-help-women-travel-safely/`; `https://blog.railyatri.in/why-smart-bus-is-safe-for-female-travelers/`

[6] `https://willerexpress.com/en/bus_search/tokyo/all/osaka/all/day_21/service_safety/`; and `https://rayhaber.com/2022/07/tcdd-yht-seyahatlerinde-cinsiyete-gore-koltuk-secimi-karari-aldi/`.

[7] Note that in Figure 1, some male-female pairs are seated next to next as the system allows for that once either of them makes a booking for both.

Figure 1: A Screenshot from the Turkish Railway Booking System

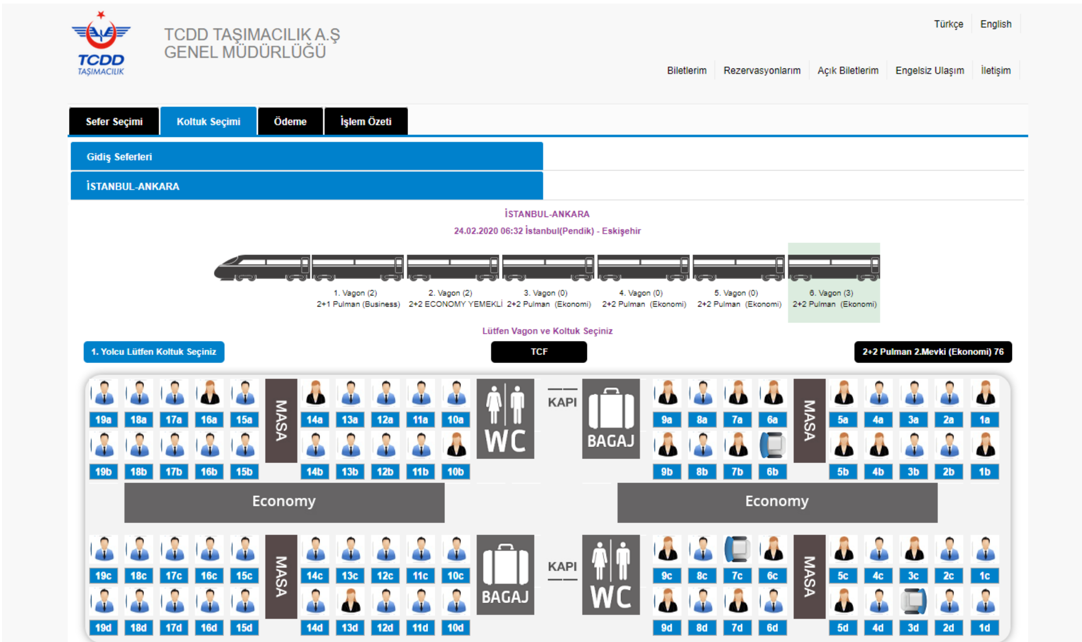
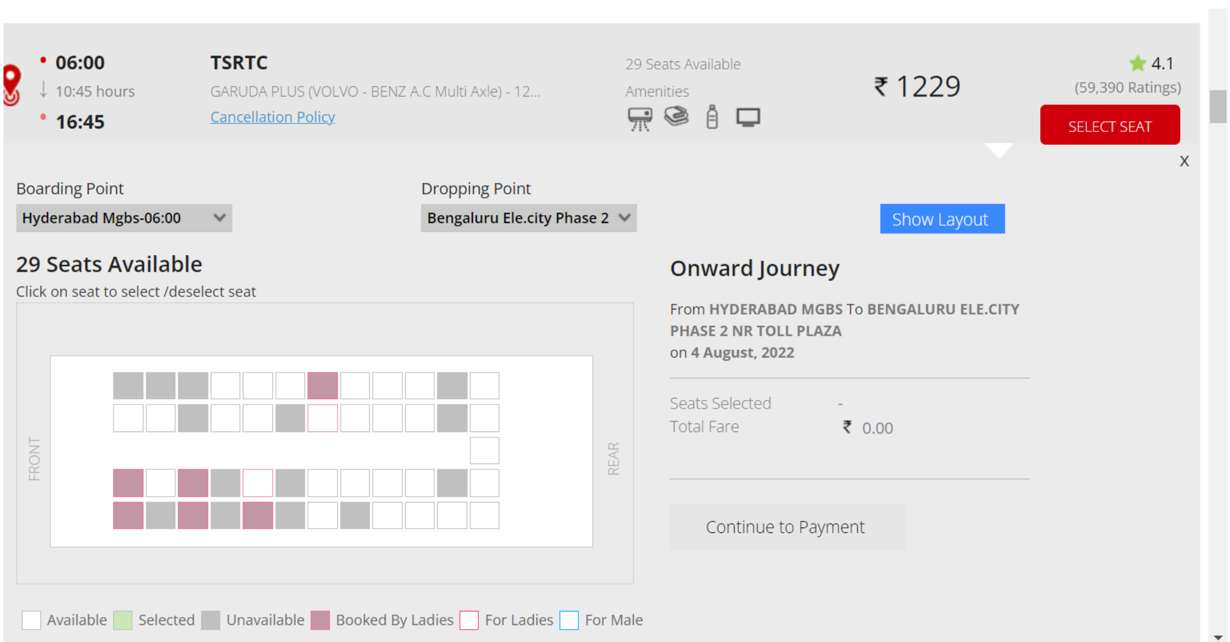


Figure 2: A Screenshot from the Indian Intercity Bus Booking System



the current first-come-first-serve based seat assignment goes beyond that in the sense that it admits avoidable waste. To see this, let us consider two pairs of seats. Suppose that the first two customers who are female choose a seat in different rows. After their bookings, the two remaining available seats can only be taken by women. If the rest of the demand only comes from men, then none of them can be seated, implying two wasted seats. However, all the seats could have been assigned in an alternative seat assignment (give the seats in either of the rows to the women) while respecting the gender-based restriction.[8] Even this simple example shows that under the current system, 50% of seats could be wasted while all could have been assigned. This flaw has caused frustration. For instance, a passenger who cannot purchase a ticket due to this restriction expressed his complaint on www.sikayetimvar.com.[9]

> *"... There are about 30 vacant seats on the train, but I cannot reserve because there is a woman next to every vacant seat, and the system does not allow me to reserve one. There cannot be such a system where there is a vacant seat on the train, but I can't get it. I have been calling the help center, but nobody helps. Isn't it unfair?"*

Another frustrated passenger expressed her experience as follows:

> *"Although there are 20 vacancies when I move to the next stage of online reservation, I cannot find a seat as a woman because there are always men sitting next to the vacant seats, the system prevents me from choosing a seat."*[10]

In each of these quoted cases above, the problem is that passengers first attempt to book a seat whose adjacent is empty, exacerbating the gender-based restrictions. Moreover, this seat-wasting issue is a concern for both the demand and supply sides as it causes economic loss. Here is an excerpt from a bus company owner in Sri Lanka:

> *"Sometimes, the seat next to a 'lady seat' has to remain empty if there is no*

[8] We could face *fairness* issues even when more than two female customers follow the two male customers.

[9] www.sikayetimvar.com is a website where people can express their complaints about the companies and brands in Turkey. See https://www.sikayetvar.com/tcdd/tcdd-hizli-tren-rezervasyon-sorunu.

[10] See https://www.sikayetvar.com/tcdd/erkek.

> *female passenger who selects it, and that's a loss of income for the bus company owners."*[11]

Of course, eliminating GBS is always an option to solve the seat-wasting issue, but women benefit from them as they feel secure. Here is an excerpt revealing it.

> *"She [Revathy] says she's been able to avoid inappropriate touching from male passengers thanks to BusSeat.lk, an online bus-booking service that gives female riders the option to book a "lady seat" on long-distance trips. With this feature, Revathy and other female travelers can make sure that the seat or seats next to them are only occupied by women, at no extra cost."*[12]

Besides the avoidable seat-waste problem of the current system, it has some other severe handicaps on fairness, efficiency, and strategic grounds, as formally shown later. Therefore, the optimal seat allocation mechanism under this restriction is a worthy research avenue. To this end, for the first time in the literature, we formulate a seat assignment problem under the gender-based restriction that no pair of passengers of different genders can be seated next to each other. More specifically, we consider a problem with a set of seats to be allocated to passengers. Seats are grouped in rows, each including either two or three seats.[13] Passengers have preferences over the seats and are prioritized based on their booking time, where the earlier reservation entails a higher priority. Each passenger is either male or female, and no pair of male-female passengers can be seated in adjacent seats.

A critical aspect of the modeling is that passengers' welfare is affected by the others' assignments. Namely, we assume that passengers always prefer having more empty seats in their assigned rows, and only when it is the same for two separate seats, they prefer the matching assigning their favorite seat among them.

[11] `https://globalpressjournal.com/asia/sri_lanka/sri-lankan-entrepreneur-develops-lady-seat-option-help-women-travel-safely/`

[12] `https://globalpressjournal.com/asia/sri_lanka/sri-lankan-entrepreneur-develops-lady-seat-option-help-women-travel-safely/`

[13] We choose this setting to cover the seating schemes in various public transportation.

We introduce a stability notion ensuring that (i) no unseated passenger can be seated without taking away the assigned seat of some higher priority passenger, and (ii) no seated passenger can propose an alternative seat assignment under which s/he is better off without harming anyone with a higher priority and the seated passengers remain the same. The first condition guarantees that passengers should be able to book so long as no higher priority passenger must be unseated for it to be possible. In other words, it advocates that passengers' inability to book should not be due to the assigned seats for earlier bookings. The second condition, on the other hand, ensures that those with earlier booking times are favorably treated in the distribution of seats. In a nutshell, our stability condition eliminates the waste (by assigning as many passengers as possible) and controls the externalities caused by the lower priority passengers' assignments on the higher priority passengers' assignments.

We refer to the aforementioned first-come-first-serve based booking as "*Myopic Serial Dictatorship*" (MSD). We have already discussed that MSD yields avoidable waste. Besides, MSD is not stable, strategy-proof, or efficient.

Given the serious deficiencies of MSD, we next introduce a mechanism called "*Adaptive Serial Dictatorship*" (ASD). In ASD, passengers are assigned their favorite seat from a constraint set one by one in order of their booking time. The novel aspect here is the constraint set each passenger encounters. ASD dynamically constructs it depending on the previous seat assignments. We show that ASD is stable, efficient, and strategy-proof. It is not maximal in the sense that one can improve ASD in terms of the number of assigned seats. Nonetheless, in contrast to MSD, ASD cannot be improvable on this basis by another stable mechanism. We indeed show that it is the only stable mechanism, automatically implying it. We also quantify the cost of stability in terms of waste and find that stability causes at most one unfilled seat that could have been taken otherwise. In any problem, a maximal matching can assign at most one more seat than the ASD's outcome. This quantification matters for the supply side as they might be concerned with the economic cost of stability, which is shown not to be significant. Moreover, under ASD, no passenger would prefer to book later. Hence, it gives passengers the incentive to make their reservations earlier.

While gender-based seating is in charge in some countries, as discussed above, it is still common to have a gender-free booking. Examples of such are abundant, but a drastic one is that Southwest Airlines does not assign seats a priori. Instead, passengers select their seats after entering the plane without any restriction. Fortunately, we can easily adapt our model and the mechanisms to the restriction-free case. In terms of results, while MSD becomes maximal, its other negative properties above still hold. ASD, on the other hand, maintains all its positive properties above, and it becomes maximal.

Although our theoretical results show MSD performs superior to ASD in all dimensions we consider, they do not quantify the possible gains that can be achieved from replacing ASD with MSD. We use simulations to measure the gain from replacing MSD with ASD regarding the number of assigned passengers and elimination of stability violations.

In Section 6, we extensively elaborate on the practical implementation of ASD and some model extensions, including couples and weak preferences. Let us emphasize that the applicability of the theoretical framework is not limited to the seat allocation problem. For instance, during the COVID-19 pandemic, in order to maintain social distancing, many airlines blocked the middle seat in planes with 3+3 seating configurations.[14] Moreover, airlines were considering implementing vaccination passports.[15] Our model can be applied to allocate seats under vaccination passport restrictions where a pair of unvaccinated passengers cannot be seated next to each other.

[14]See https://www.cnn.com/2021/04/14/health/airplane-seating-covid-risk-cdc-study-wellness/index.html, https://news.delta.com/delta-extends-middle-seat-blocking-through-april-2021-only-us-airline-continue-providing-more-space, https://www.forbes.com/sites/advisor/2020/12/07/master-list-of-us-airline-seating-and-mask-covid-19-policies/?sh=4aee98f11bb4.

[15]See https://www.nbcnews.com/business/travel/next-frontier-air-travel-digital-passports-proof-vaccination-n1261338.

2 Related Literature

Matching theory has been applied to many real-life markets, including school choice (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu et al., 2009; Kesten, 2010), organ exchange (Roth et al., 2004; Ergin et al., 2017), refugee resettlement (Trapp et al., 2020), and cadet-branch matching (Sönmez and Switzer, 2012). To the best of our knowledge, this paper is one of the first applications of matching theory to the seat allocation problem in public transportation. While there is no closely related paper, Our model exhibits externalities,[16] hence, it is generally related to the matching with externalities literature. Sasaki and Toda (1996) define a stability notion where the blocking agents consider all possible reactions from others. They show a stable matching always exists whenever agents consider the worst-case scenarios. Fisher and Hafalir (2016) consider a one-to-one matching problem where agents ignore the externalities caused by their actions. They find some conditions under which a stable matching always exists. Hafalir (2008) formulates a marriage problem with endogenous beliefs as to the others' reactions to a blocking pair. He comes up with a particular belief formation with which the existence of a stable matching is guaranteed. Bando (2012) consider a many-to-one matching setting where only the firm preferences exhibit externalities and introduce a stability notion eliminating further deviations within a blocking coalition. The author identifies some conditions ensuring the existence of a stable matching. In the same setting, Bando (2014) introduces a deferred-acceptance-based mechanism to find a stable matching.

Pycia and Yenmez (2022) extend the classical substitutes condition to a two-sided matching with externalities setup and obtain it as a sufficient and necessary (in the maximal domain sense) condition for the existence of a stable matching.

The key conditions for the existence of a stable matching in Pycia and Yenmez (2022), Bando (2012), and Fisher and Hafalir (2016) do not hold in our setting (see Footnote 23). However, we still obtain existence because of our stability definition. Their stability concept

[16] Passengers' welfare is affected by the others' seat assignments.

is an equilibrium notion, eliminating any beneficial reassignments. On the other hand, ours is a fairness concept ruling out any reassignment proposal by an agent benefiting herself while not hurting anyone with a higher priority than the former as long as any avoidable seat wasting is addressed.

In a general framework covering both transferable and non-transferable utility cases, Rostek and Yoder (2020) consider agents' matching through contracts. They introduce a complementarity condition ensuring that a selected contract is never rejected after new contracts become available. Under this condition, Rostek and Yoder (2020) obtain that a stable matching always uniquely exits. We also obtain the uniqueness of stable matching, however, their condition does not hold in our setup.[17] Echenique and Yenmez (2007) consider a college assignment problem where students are interested in not only the colleges they are assigned to but also their peers at their colleges. They propose a fixed-point algorithm to find a core assignment whenever it exists. In a coalition formation setting, Pycia (2012) allows for peer effect and complementarities and identifies a condition to guarantee a core assignment exists.

In addition to externalities, the other key feature of our model is the gender-based constraint over assignments. There is an extensive literature on matching with constraints. Kamada and Kojima (2015) consider a two-sided matching problem where hospitals are partitioned into regions, and regions have caps to restrict the number of doctors assigned to these regions. In the same setup, Kamada and Kojima (2016) address a more general class of constraints restricting the number of assigned doctors at each hospital depending on those at the other hospitals. Note that this class contains the constraints in Kamada and Kojima (2015). They both introduce stability notions and pursue mechanism designs. Note that none of these constraint sets include ours, as genders matter to ours. On the other hand, Kamada and Kojima (2020) consider a class of constraints restricting the group of doctors each hospital can be assigned a priori. While the doctors' identities play a role here, the

[17] For instance, whenever only one available seat is left, a passenger chooses it. However, she may very well not select it whenever there are other available seats as well.

constraints are hospital specific. This is not the case in our setting as the constraint on each seat (hospital with a unit capacity) depends on the gender of the passenger seated at its adjacent. Kamada and Kojima (2020) characterize the subclass of their constraint set under which a doctor-optimal stable matching exists. Apart from the differences between the constraint classes covered in these papers and ours, none of them exhibit externalities.

There is a body of literature on the seat allocation problem but from a completely different perspective from ours. This literature mainly studies how to increase revenue from seat sales. Yuan and Nie (2020) investigate how seat-grouping in trains should be done based on consumer behavior to increase revenue. Sawaki (1989) considers a price discrimination model and finds the optimal number of seats that should be sold for a low fare to maximize the expected revenue. Freisleben and Gleichmann (1993) study overbooking predictions to decrease the empty seats in flights.

The last paper we want to touch on is Kominers and Sönmez (2016). They introduce a new class of matching problems, capturing the airline-seat upgrade as its application. This problem is entirely different from ours. Instead of considering seat assignments, it addresses the means (cash payment, elite status, or miles) of seat-class upgrade and shows that Kominers and Sönmez (2016)'s model can be applied to assigning agents' upgrade requests to these means.

3 Model

In this section, we first introduce the **seat allocation problem**. Then, we provide the axioms used in our analysis.

3.1 Seat Allocation Problem

Let $(N, S, \rhd, \succ)$ be a seat-allocation problem described below.

- N and S are the non-empty sets of **agents** (passengers) and **seats**, respectively.

- Each agent is either **male or female**.[18] Let N^m and N^f be the sets of male and female agents, respectively.
- $\succ$ is the **priority ordering** over the agents such that the earlier an agent makes a reservation, the higher priority s/he has. We write $i \succ j$ if agent i has a higher priority than agent j.
- Each agent i has a **strict ranking** $\rhd_i$ over S. Let $\rhd = (\rhd_i)_{i \in N}$ be the ranking profile.

Let $N = \{i_1, .., i_n\}$ be the enumeration of the agents such that for each $k < k'$, $i_k \succ i_{k'}$.[19] Let $U(i_k) = \{i_{k'} : k' < k\}$, that is, the set of agents who come earlier than i_k in the ordering. Seats are grouped in rows consisting of either 2 or 3 seats.[20] Let $r \geq 1$ be the total number of rows. We write σ_s for the type of seat s. In the case of 2-seat rows, $\sigma_s \in \{1, .., r\} \times \{W, A\}$; and otherwise, $\sigma_s \in \{1, .., r\} \times \{W, M, A\}$. Its first and second components denote the row and side of seat s where W, M, A stand for window, middle, and aisle sides, respectively. Notice that every seat has a unique type.

We say that a seat is **adjacent** of another if they are in the same row and next to each other. For instance, in the case of 2-seat rows, seats in the same row are adjacent to each other. In the case of 3-seat rows, each seat-pair in a row except the window-aisle one is adjacent to each other. Let τ_s be the seats that are in the same row as seat s, including seat s. Note that $|\tau_s|$ gives us the number of seats each row contains.

Regarding the agents' seat rankings, we assume that in the case of 3-seat rows, each agent ranks the middle seat below the two other seats in the same row.

Assumption 1. *In the case of* 3*-seat rows, for each agent i and pair of seats $s, s' \in \tau_{s'}$ where $\sigma_{s'} = (r, M)$, $s \rhd_i s'$.*

[18]These types may differ in other applications.

[19]In the rest of the paper, we use this ordering unless otherwise stated.

[20]It is possible to extend our analysis to capture more than 3 seats in a row. However, since different seating possibilities need to be considered, the cost of such an extension will be having more complex algorithms. Moreover, our modeling choice can cover almost all applications of gender-based seating restrictions.

Note that this assumption allows agents to prefer the aisle side in some rows but the window side in others. We need Assumption 1 for ASD to satisfy the gender-based restriction in the case of 3-seat rows (see Footnote 37 for details). That is, all our results related to case with 2-seat rows hold without Assumption 1. A **matching** μ is an assignment of seats to agents such that each agent receives at most one seat, and no seat is assigned to more than one agent. For any $k \in N \cup S$, we write μ_k for the assignment of k under μ. We write $\mu_k = \emptyset$ if agent (seat) k does not receive a seat (is not assigned to an agent). Let $\mu_S = \{i \in N : \mu_i \neq \emptyset\}$. Under the gender-based seating restriction, a matching μ is **feasible** if there do not exist $i \in N^m$ and $j \in N^f$ such that μ_i and μ_j are adjacent seats. When there is no gender-based seating restriction, any matching is feasible. In the rest of the paper, we only consider feasible matchings; for ease of exposition, we refer to them as matching.

An agent's well-being at a matching depends not only on his seat assignment per se, but also the availability statuses of the other seats in the same row. In other words, the problem exhibits externalities. Hence, agents' ordering over the seats do not fully specify their preferences over matchings. Here, we assume a particular class of externalities, where each agent always prefers having the other seats in his assigned row be empty. Section 6 will consider general externalities and how the analysis could be carried out. But, in the benchmark model, we have the following suppositions regarding the preferences.

Assumption 2. *An agent i prefers matching μ to ν if either*

(i) $\mu_i \in S$ *and* $\nu_i = \emptyset$, *or*

(ii) $\mu_i, \nu_i \in S$ *and* $|\{s'' \in \tau_{\mu_i} : \mu_{s''} = \emptyset\}| > |\{s'' \in \tau_{\nu_i} : \nu_{s''} = \emptyset\}|$, *or*

(iii) $\mu_i, \nu_i \in S$; $|\{s'' \in \tau_{\mu_i} : \mu_{s''} = \emptyset\}| = |\{s'' \in \tau_{\nu_i} : \nu_{s''} = \emptyset\}|$; *and at least one adjacent seat of* μ_i *is empty while none of the adjacent seats of* ν_i *are empty, or*

(iv) $\mu_i, \nu_i \in S$ *and none of the first three cases holds and* $\mu_i \rhd_i \nu_i$.

In other words, the first condition assures that each agent always prefers receiving a seat. The second condition says that each agent always prefers having more empty seats in his assigned row. In the case of the same number of empty seats in his row, he prefers having his adjacent seats empty—Condition (iii).[21] If none of these holds, only then do his preferences come from his ranking over the seats. Note that we do not impose any preferential supposition over pairs of matchings where an agent receives the same middle seat in both cases, but a different adjacent seat is empty in each case. Agents can be indifferent or have strict preferences between such two matchings.[22] For the 2-seat row case, Assumption (*iii*) is void.[23]

Let R_i denote the agent i's preferences over matchings. We write P_i for its strict part. Two notes are in order: (*i*) each $\triangleright_i$ induces different preferences, and (*ii*) agents do not have preferences over the other agents, implying that they are all indifferent between matchings that differ in terms of the identities in their rows. In the rest of the paper, we fix all the primitives except the agents' ranking over the seats and denote the problem by $\triangleright$.

3.2 Axioms

Next, we define the axioms used in our analysis. We start with our stability notion.

[21]To see this case, consider a 3-seat row with two agents, say i, j. Let us consider two seating arrangements: (1) Agents i and j are seated at the window and aisle sides, respectively; and (2) Agents i and j are seated at the window and middle sides, respectively. In both cases, only one seat is left empty. However, both agents prefer the first arrangement as their adjacent seats are empty.

[22]This case does not matter in our solution, as, in line with Assumption 1, it always leaves the middle seats empty to the extent possible.

[23] Pycia and Yenmez (2022) introduce a monotone externality condition imposing that an agent never accepts a previously rejected alternative after the others have an at least weakly better alternative. This condition proves to be key for their whole analysis as well as the existence of a stable matching. This condition fails to hold in our setting. To see this, let $N = \{i_1, i_2\}$, $S = \{s_1, s_2, s_3, s_4\}$, $\tau_{s_1} = \{s_1, s_2\}$ and $\tau_{s_3} = \{s_3, s_4\}$. Suppose that both agents' preferences follow s_1, s_3, s_2, s_4. Consider μ and μ' where $\mu_{i_2} = \emptyset$ and $\mu'_{i_2} = s_2$. From $\{s_1, s_3, s_4\}$, agent i_1 chooses s_1 given μ (that is, whenever agent i_2 is unseated), whereas she selects s_3 under μ'. Thus, she starts choosing the rejected s_3 after i_2 has a better alternative, violating the monotone externality condition. Bando (2012) proposes a similar condition, called positive externality, and obtains the existence of a stable matching under it. This condition says that a firm is better off whenever other firms hire new workers. From the example above, we see that i_1 would receive the best assignment for herself whenever i_2 remains unassigned, violating positive externality. Fisher and Hafalir (2016) consider a one-to-one matching problem with cardinal utilities where externalities do not matter a lot. While we do not have cardinal payoffs, the assignments of the others easily change the selected seat of an agent. Hence it does not hold in the current problem.

Definition 1. *A matching μ is **stable** if, for each agent i_k, (a) whenever $\mu_{i_k} = \emptyset$, there is no matching μ' where $\mu'_{i_k} \neq \emptyset$ and $U(i_k) \cap \mu_S \subseteq U(i_k) \cap \mu'_S$, and (b) whenever $\mu_{i_k} \neq \emptyset$, there is no matching μ' where $\mu'_S = \mu_S$, $\mu' \; P_{i_k} \; \mu$, and for each $j \in U(i_k)$, $\mu' \; R_j \; \mu$.*

Our stability notion is different from its usual definition in standard object assignment problems because of the externalities in the seat allocation problem and targeting the elimination of waste. Condition (a) ensures that no seat is wasted (see Remark 1 below for details), and no agent is unassigned for the sake of a lower priority agent. Condition (b), on the other hand, imposes that no agent can be better off without harming a higher priority one or causing someone to be unseated.

Remark 1. *The stability of a matching μ implies that for each agent i with $\mu_i = \emptyset$, there exists no matching μ' where $\mu'_S = \mu_S \cup \{i\}$. No agent can receive a seat without creating a newly unassigned agent. In other words, no seat is wasted, which is a property known as non-wastefulness, implied by stability.*

Matching μ **Pareto dominates** μ' if for each agent $i \in N$, $\mu \; R_i \; \mu'$, where this holds strictly for some agent. Matching μ is **efficient** if it is not Pareto dominated by another matching. A matching μ is **maximal** if there does not exist another matching ν such that $|\mu_S| < |\nu_S|$, i.e., no matching allocates more seats than μ.

A **mechanism** ψ is a systematic way to produce a matching for each problem $\rhd$. We write $\psi(\rhd)$ to denote the outcome of ψ at problem $\rhd$. Mechanism ψ is $<$ *stable, efficient, maximal* $>$ if $\psi(\rhd)$ is $<$stable, efficient, maximal $>$ for each problem $\rhd$. Mechanism ψ is **strategy-proof** if there are no problem $\rhd$ and agent i with ranking $\rhd'_i$ such that $\psi(\rhd'_i, \rhd_{-i}) \; P_i \; \psi(\rhd)$.[24]

We are now ready to provide the theoretical and computational results. After having presented them in order, we will discuss some straightforward model generalizations in the Discussion section.

[24] $\rhd_{-i}$ is the ranking profile of all the agents except agent i over the seats, i.e., $\rhd_{-i} = (\rhd_j)_{j \neq i}$.

4 Results

In what follows, we provide the results in two sections: **Restricted** and **Unrestricted** cases. In the former, no pair of agents of different genders can be seated next to each other. We impose it as a feasibility requirement over matchings and only consider those satisfying it. In the unrestricted case, however, each agent can be seated next to anyone irrespective of gender. Hence, we do not have any feasibility requirements over matchings. The following result holds for both restricted and unrestricted cases.[25]

Proposition 1. *Let μ be a stable matching in problem $\rhd$. Then, μ is efficient, i.e., stability implies efficiency.*

4.1 Restricted Case

In this section, we assume that no pair of agents of different genders can be seated next to each other. We start our analysis by first defining a mechanism, called *Myopic Serial Dictatorship*, based on the commonly used first-come-first-serve assignment.

4.1.1 Myopic Serial Dictatorship (MSD)

MSD is based on the serial dictatorship in which each agent selects his most preferred seat one by one following the ordering. We formally define the MSD mechanism below.

Myopic Serial Dictatorship:

By following the agent-ordering, the MSD mechanism selects its outcome through the following steps. For $k \in \{1, \dots, n\}$

Step k**.** Let us consider the seats in the rows including the highest number of empty seats. Among these seats, agent i_k selects the best ranked one (with respect to $\rhd_{i_k}$) whose adjacent was not taken by an agent of a different gender in a previous step. If such a seat does not exist, then i_k stays unassigned.

[25]We provide the proofs of all results, except Proposition 7, in Appendix B.

MSD terminates by the end of Step n, that is, when all agents are processed. We first illustrate how MSD works via a simple example.

Example 1. *Let* $N^m = \{m_1, m_2, m_3, m_4\}$, $N^f = \{f_1\}$, $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$, $\tau_{s_1} = \{s_1, s_2\}$, $\tau_{s_3} = \{s_3, s_4\}$, $\tau_{s_5} = \{s_5, s_6\}$, *and* $m_1 \succ m_2 \succ m_3 \succ m_4 \succ f_1$. *Agents' strict rankings over* S *are:*

$$s_1 \rhd_{m_1} s_2 \rhd_{m_1} s_3 \rhd_{m_1} s_4 \rhd_{m_1} s_5 \rhd_{m_1} s_6$$

$$s_3 \rhd_{m_2} s_2 \rhd_{m_2} s_1 \rhd_{m_2} s_4 \rhd_{m_2} s_5 \rhd_{m_2} s_6$$

$$s_3 \rhd_{m_3} s_1 \rhd_{m_3} s_2 \rhd_{m_3} s_4 \rhd_{m_3} s_5 \rhd_{m_3} s_6$$

$$s_2 \rhd_{m_4} s_1 \rhd_{m_4} s_3 \rhd_{m_4} s_4 \rhd_{m_4} s_5 \rhd_{m_4} s_6$$

$$s_2 \rhd_{f_1} s_1 \rhd_{f_1} s_3 \rhd_{f_1} s_4 \rhd_{f_1} s_5 \rhd_{f_1} s_6$$

Let us run MSD *in the problem.*

Step 1. *Since all the seats are empty, agent* m_1 *selects* s_1, *which is the best ranked seat with respect to* $\rhd_{m_1}$.

Step 2. *Agent* m_2 *selects* s_3, *which is the best ranked seat with respect to* $\rhd_{m_2}$ *in the rows with the highest number of empty seats.*

Step 3. *Agent* m_3 *selects* s_5, *which is the best ranked seat with respect to* $\rhd_{m_3}$ *in the rows with the highest number of empty seats.*

Step 4. *Since a seat in each row has already been taken in the previous steps, agent* m_4 *selects* s_2, *which is the best ranked seat with respect to* $\rhd_{m_4}$.

Step 5. *Since each row contains a man, agent* f_1 *cannot be seated.*

Hence, MSD *selects matching* μ *where* $\mu_{m_1} = s_1$, $\mu_{m_2} = s_3$, $\mu_{m_3} = s_5$, $\mu_{m_4} = s_2$, *and* $\mu_{f_1} = \emptyset$.

Below, we show that MSD fails to satisfy each of the desirable properties we have discussed.

Proposition 2. *In the restricted case, MSD is not stable, or maximal, or efficient, or strategy-proof.*

Remark 2. *MSD assigns seats sequentially, one by one, following the ordering. Even though agents select the best remaining seats in their turn, the outcome is neither stable nor efficient. This shows that no mechanism assigning seats sequentially as agents arrive can admit these properties. Thus, we need a mechanism that first gathers all the agents and then assigns the seats once and for all. Our mechanism proposal is of this type with the advantage that agents will at least know whether they receive a seat by their arrival.*

Given the negative results presented in Proposition 2, our objective is to introduce a mechanism satisfying desirable properties to the extent possible. Unfortunately, Proposition 3 shows that, in the restricted case, stability and maximality are incompatible.

Proposition 3. *In the restricted case, there does not always exist a stable and maximal matching.*

Proposition 3 implies that there does not exist a stable mechanism that is also maximal. Given the incompatibility between stability and maximality, we focus on the maximality among stable matchings. We say that a matching μ is **constrained maximal** if there is no stable matching ν with $|\mu_S| < |\nu_S|$. A mechanism is *constrained maximal* if it always produces a constrained maximal matching.

Let us revisit the example in the proof of Proposition 2 that shows the lack of maximality of MSD. There, matching ν is stable and assigns more agents than the MSD outcome, revealing that MSD is not even constrained maximal.

Proposition 4. *In the restricted case, MSD is not constrained maximal.*

The reason why MSD performs poorly is due to its inflexibility. It is a greedy algorithm: Once an agent selects a seat, it becomes permanent. However, because of externalities, the agent's well-being depends on later selections, undermining its efficiency and stability.

Likewise, agents select their seats without considering the restrictions they impose on the allocation of their adjacent seats, owing to gender-based seating. Because of the greedy nature of MSD, the problematic seat selections, causing avoidable waste, cannot be fixed in the mechanism. Thus, to fix all these, we need a mechanism that addresses both externalities and gender-based restrictions in its dynamics and assigns the best possible seats to the agents in order of their priorities. We will introduce such a mechanism in the next section.

4.1.2 Adaptive Serial Dictatorship

Given the serious handicaps of MSD, we introduce a new mechanism, which will be shown to be superior to MSD in many aspects. We call this mechanism *Adaptive Serial Dictatorship* and denote it by ASD. For the sake of expositional simplicity, we define ASD below when there are two seats in each row. In Appendix A, we provide its general definition, encapsulating both 2-seat and 3-seat cases. The proofs are all given for this general version. Hence, all the results hold for both cases.

Adaptive Serial Dictatorship

Step 1. We first tentatively determine who will be seated. To this end, by following the agent-ordering, we apply the following steps one by one for each agent. For $k \in \{1, .., n\}$,

Substep $1.k$. Let us consider agent i_k. If there is an available seat such that the adjacent seat is taken by an agent of the same gender as i_k, then let i_k receive the best one (with respect to $\triangleright_{i_k}$) among such seats. Otherwise, if there is an empty row, then let i_k be seated at his/her favorite seat among those in the empty rows. If none of these hold, then let i_k be unassigned.

This procedure terminates by the end of Substep $1.n$.

Let μ^0 be the tentative matching attained at the end of Step 1. We exclude all the agents in $N \setminus \mu^0_S$ from the problem, and they become permanently unassigned. Let us displace the rest from their assignments under μ^0, and each seat becomes available.

Let c be the total number of empty rows under μ^0. If $c > 0$, then we go to Step 2.[26] Otherwise, we go to Step 3.

Step 2. We only consider the top c agents in the agent-ordering. We first let i_1 pick his/her favorite available seat. We remove the agent and the selected row. Among the remaining seats, we repeat the same procedure one by one following the agent-ordering, and it ends after the selection of i_c. We then go to Step 1 in the reduced problem with the remaining agents and seats.

Step 3. We have the following exhaustive cases.

Case 1: ***All seats are taken.*** We go to Step 4.

Case 2: ***There is at least one row where only one seat is taken.*** There can be at most two rows where only one seat is taken. We have the following subcases.

Subcase 2.1: ***There are two rows where only one seat is taken.*** By Step 1's definition, one of these rows contains a woman, and the other one contains a man. Let us suppose that the top agent under the ordering is a man. The other case follows from the symmetric argument. We let the top agent, who is a man, choose his best seat and remove him along with his row from the problem. We apply Step 4 until the top woman takes her turn. When it is the top woman's turn, she selects the best remaining seat in a completely empty row. We remove her, the selected seat, and its row from the problem. We then go to Step 4.

Subcase 2.2: ***Only one row containing one seated agent whose gender is the same as the top agent.*** We let the top agent choose her/his best remaining seat and remove the selected seat and its row from the problem. We then go to Step 4.

Subcase 2.3: ***Only one row containing one seated agent whose gender is different from the top agent.*** Let agent j be the top agent of the same gender as the agent whose row has an empty seat. We apply Step 4 till agent j. Whenever it is agent j's turn, we let him/her choose his/her best seat in a completely empty row. We remove the

[26]Notice that, if $c > 0$, then $N = \mu^0_S$.

row and go to Step 4.

Step 4. By considering the remaining seats and unassigned agents, one by one following the agent-ordering, we do the following. We start with the first agent (with respect to the agent-ordering) and let him/her choose his/her best available seat. Let us then consider the second agent. Suppose she is a woman (the other case follows from a symmetric argument). Let us calculate the total number of empty seats in the rows where a woman has already been seated. If this number is equal to the number of unseated women, then we let her choose the best seat in a row where a woman has already been seated. Otherwise,[27] she chooses her best seat in rows where no man has already been seated. We continue in the same manner until the last agent.

The algorithm terminates by the end of Step 4. The assignments obtained by the above steps define the outcome of the algorithm. We run *ASD* in a problem below.

Example 2. *We consider the same problem given in Example 1. ASD selects its outcome through the following steps.*

SubStep 1.1***.*** *Since all seats are empty, m_1 receives s_1, which is the best ranked seat under $\triangleright_{m_1}$.*

SubStep 1.2***.*** *Since s_2 is the only seat whose adjacent seat has been already taken by a man, m_2 receives s_2.*

SubStep 1.3***.*** *Since there is no seat with an adjacent seat is taken by a man, m_3 receives s_3, which is the best ranked available seat under $\triangleright_{m_3}$.*

SubStep 1.4***.*** *Since s_4 is the only empty seat whose adjacent seat has been already taken by a man, m_4 receives s_4.*

SubStep 1.5***.*** *Since there is no seat whose adjacent seat is taken by a woman, f_1 receives s_5, which is the best ranked available seat under $\triangleright_{f_1}$.*

Let μ^0 be the tentative matching attained at the end of Step 1. There is no empty row under μ^0. Hence $c = 0$, and we continue with Step 3.

[27]This number cannot exceed the number of unseated women.

Step 3. *Since only one row contains one seated agent whose gender is different from the top agent, we consider Subcase 2.3 and apply Step 4 as follows.*

Step 4. *Agents* m_1, m_2, m_3, *and* m_4 *choose* s_1, s_3, s_2, *and* s_4, *respectively. Agent* f_1 *chooses* s_5 *and the algorithm terminates. Hence, ASD selects matching* ν *where* $\nu_{m_1} = s_1$, $\nu_{m_2} = s_3$, $\nu_{m_3} = s_2$, $\nu_{m_4} = s_4$, *and* $\nu_{f_1} = s_5$.

In its first step, ASD identifies the maximal set of agents that can be seated by following the agent-ordering. This step causes ASD not to waste an additional seat beyond what is caused by stability and gender-based restriction. Agents' permanent assignments are determined in its other steps. By taking care of the agents' preferences, including externalities, ASD assigns the best possible seats to the agents by respecting their priorities.

Remark 3. *Before moving to its allocative and strategic properties, let us first note that ASD runs in a polynomial time in both agents and seats. If we add one more agent, it causes at most one additional step to run. On the other hand, adding a new row of seats entails at most* 2 *additional steps.*

We are now ready to study the properties of ASD. Theorem 1 shows that ASD satisfies stability, efficiency, and strategy-proofness.

Theorem 1. *In the restricted case, ASD is stable, efficient, and strategy-proof.*

One can wonder whether there exists another stable mechanism. Theorem 2 shows that in any problem there exists a unique stable matching, and therefore ASD is the unique stable mechanism.

Theorem 2. *In the restricted case, ASD is the unique stable mechanism.*

The following corollary is immediately implied by Proposition 3 and Theorem 2.

Corollary 1. *In the restricted case, ASD is not maximal. However, it is constrained maximal.*

Fortunately, stability does not cause too many empty seats that otherwise would be filled: In any problem, a maximal matching can assign at most one more seat than the ASD's outcome.

Proposition 5. *In the restricted case, let μ' be a maximal matching at a problem $\rhd$. Then, $|\mu'_S| \leq |ASD_S(\rhd)| + 1$.*

Recall that the earlier a booking an agent makes, the earlier he comes in the ordering. Thus, an interesting question is whether agents prefer to make a booking earlier. To address this question, we start including $\succ$ in the problem notation and write $(\rhd, \succ)$ instead of $\rhd$.

We say that $\succ'$ is an **improvement** over $\succ$ for agent i if for each $j, k \in N \setminus \{i\}$, $i \succ j$ implies $i \succ' j$, and $j \succ' k$ if and only if $j \succ k$. Mechanism ψ **respects improvements** if there is no problem $(\rhd, \succ)$ and $\succ'$ such that $\succ'$ is an improvement over $\succ$ for agent i, and $\psi(\rhd, \succ) \; P_i \; \psi(\rhd, \succ')$.

Theorem 3. *In the restricted case, ASD respects improvements.*

Hence, under ASD, agents are incentivized to book their reservations earlier.

4.2 The Unrestricted Case

In this section, we assume that there is no gender-based restriction in the sense that an agent can be seated next to any agent independent of their gender. We start our analysis by first adapting the *Myopic Serial Dictatorship* to the unrestricted case below.

4.2.1 Myopic Serial Dictatorship (MSD)

When there is no gender-based restriction, we can apply the MSD mechanism defined in Section 4.1.1 by considering that all agents are of the same gender. For the sake of completeness, we define the MSD mechanism in the unrestricted case.

Myopic Serial Dictatorship in the Unrestricted Case

By following the agent-ordering, the MSD mechanism selects its outcome through the following steps. For $k \in \{1, \ldots, n\}$

Step k**.** Let us consider the set of rows with the highest number of empty seats. Among the seats in these rows, agent i_k selects the best ranked one (with respect to $\rhd_{i_k}$). If such a seat does not exist, then i_k stays unassigned.

MSD terminates by the end of Step n. We show below that in the unrestricted case, while MSD becomes maximal, it does not satisfy the other desirable properties.

Proposition 6. *In the unrestricted case, MSD is maximal. However, it is not stable, efficient, or strategy-proof.*

Given that MSD does not satisfy most of the desirable properties, we straightforwardly adapt ASD to the unrestricted case.

4.2.2 Adaptive Serial Dictatorship (ASD)

As explained for MSD, we can apply the ASD mechanism defined in Section 4.1.2 by supposing that all the agents are of the same gender. For the sake of completeness, we define the ASD mechanism in the unrestricted case. As the same as before, below defines ASD for the 2-seat case. Its general definition is given in Appendix A. The result in this section holds for the general version, capturing both 2-seat and 3-seat rows.

Adaptive Serial Dictatorship in the Unrestricted Case

Step 1. We first tentatively allocate seats among agents. To this end, by following the agent-ordering, we apply the following steps one by one for each agent. For $k \in \{1, .., n\}$,

SubStep 1.k. Let us consider agent i_k. If there is an available seat whose adjacent seat has already been taken, then let i_k receive his/her favorite seat among such seats. Otherwise, if there is an empty row, then let i_k be seated at his/her favorite seat among the ones in the empty rows. If none of these hold, then let i_k be unseated.

This procedure terminates by the end of Substep 1.n. Let μ^0 be the matching at the end of Step 1. We exclude all the agents in $N \setminus \mu^0_S$ from the problem and let them be permanently unassigned.

Let c be the total number of empty rows under μ^0. Let us displace the agents from their assignments under μ^0, and each seat becomes available to be assigned. If $c > 0$, then we go to Step 2.[28] Otherwise, we go to Step 3.

Step 2. We only consider the top c agents in the agent-ordering. We first let i_1 pick his/her favorite seat. We remove the agent and the selected row. Among the remaining seats, we repeat the same procedure one by one following the agent-ordering, and it ends after the selection of i_c. We then go to Step 1 in the reduced problem, where only the remaining agents and seats are considered.

Step 3. We have the following exhaustive cases.

Case 1: ***All seats are taken.*** We go to Step 4.

Case 2: ***There is a row with only one seat is taken.*** No other row contains an empty seat. We let the top agent choose her/his best remaining seat. We remove the agent along with the selected row and go to Step 4.

Step 4. In the reduced problem, we let each agent choose his/her best remaining seat one by one following the agent-ordering.

The algorithm terminates by the end of Step 4. The assignments obtained in the course of the above steps define the outcome of the algorithm.

Note that ASD in the unrestricted case is equivalent to the restricted case's ASD whenever all the agents are of the same gender. Therefore, all the earlier positive properties of ASD carry over to the unrestricted case. Moreover, an agent fails to receive a seat only when all seats are already taken, implying that ASD becomes maximal in the unrestricted case as well.

Proposition 7. *In the unrestricted case, ASD is maximal, stable, efficient, strategy-proof, and respects improvements. Moreover, it is the unique stable mechanism.*

For the sake of brevity, we refer to the related results under the restricted case and skip

[28] If $c > 0$, then $N = \mu^0_S$.

the proof of Proposition 7.

5 Simulations

This section uses computer simulations to measure possible gains from replacing the first-come-first-served based current procedure with ASD. In particular, we calculate the fractions of agents assigned to a seat under the current procedure and ASD under various scenarios based on the number of agents, correlation in preferences over seats, and gender distributions. Moreover, we also calculate the number of instances in which priorities are violated when there is no restriction on the seating. Here, we take MSD as a proxy for the current procedure.

We run separate simulations for 2-seat and 3-seat cases. Under both cases, $|N|$ agents[29] are ordered according to their booking times, and no two agents book at the same time. Hence, we have a strict priority order over the agents. Instead of randomly choosing the gender of each agent with the same probabilities, we consider different distributions in which the first half of the agents and the second half of the agents have the same probability of being female and male, respectively. For the first half of the agents, an agent is a female (male) with probability $\delta \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$ $(1-\delta)$, and for the remaining half of the agents, an agent is female (male) with probability $1-\delta$ (δ). Hence, we aim to have a population with equal shares of females and males.

To construct the preference of each agent i, we calculate her/his utility from being assigned to each seat s as follows:

$$U_{i,s} = \alpha \times C_s + (1-\alpha) \times D_{i,s},$$

where $C_s \in (0,1)$ represents the common utility received by all individuals from seat s and $D_{i,s} \in (0,1)$ represents the individual specific utility received by agent i from seat s. Both

[29] We take $|N|$ as an even number.

C_s and $D_{i,s}$ are selected from i.i.d. standard uniform distribution. The correlation between the preferences of the agents is captured by variable $\alpha \in \{0, 0.25, 0.5, 0.75, 1\}$. The higher the α is, the more correlated preferences are. The calculated utility values of agents over the seats are used to construct the ordinal preferences of agents over the seats.

For 2-seat case, we set the number of rows to 50 and the number of seats to 100. We consider five different cases based on the number of agents, namely 80, 90, 100, 110, and 120. When the number of agents is less than the number of seats, there is less competition for the seats, and, in the unrestricted case, every agent can be seated. On the other hand, when the number of agents is more than the number of seats, the competition is more fierce, and some agent has to be unassigned. Our theoretical results imply that when the number of agents is 80 or 90, ASD assigns each agent to a seat. Moreover, ASD will waste at most one seat for the remaining cases. Our simulations verify these theoretical results. On the other hand, under MSD, we observe many wasted seats, specifically for a lower level of δ. The MSD performs poorly when the number of agents is less than the number of seats. This follows from two key facts: (1) when there are fewer agents than seats, there is less competition for the seats, and agents are not seated due to the skewed distribution of bookings, and (2) when there are more agents than seats, we can find enough agents to fill them. We also observe that α does not affect the ratio of seated agents under MSD to that under ASD. We present our simulation results for the 2-seat case for $\alpha = 0.5$ in Figure 3.

For the 3-seat case, we set the number of rows as 40 and the number of seats as 120. We consider five different cases based on the number of agents, namely 100, 110, 120, 130, and 140. Our theoretical results imply that for the cases of 100 and 110 agents, ASD can seat all agents independent of the gender distribution. Moreover, for the remaining cases, under ASD, there will be at most one unfilled seat. Our simulations verify these theoretical results. On the other hand, under MSD, we observe many wasted seats, specifically for the lower level of δ, independent of the number of agents. We also observe that α does not affect the ratio of seated agents under MSD to that under ASD. We represent our simulation

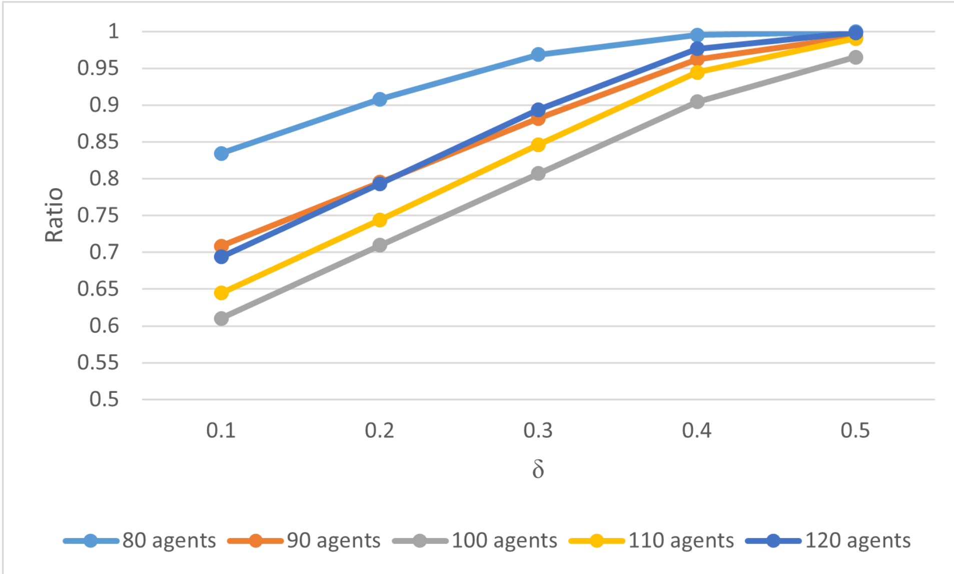


Figure 3: Ratio of seated passengers under MSD and ASD (2-seat restricted case)

results for the 3-seat case for $\alpha = 0.5$ in Figure 4.

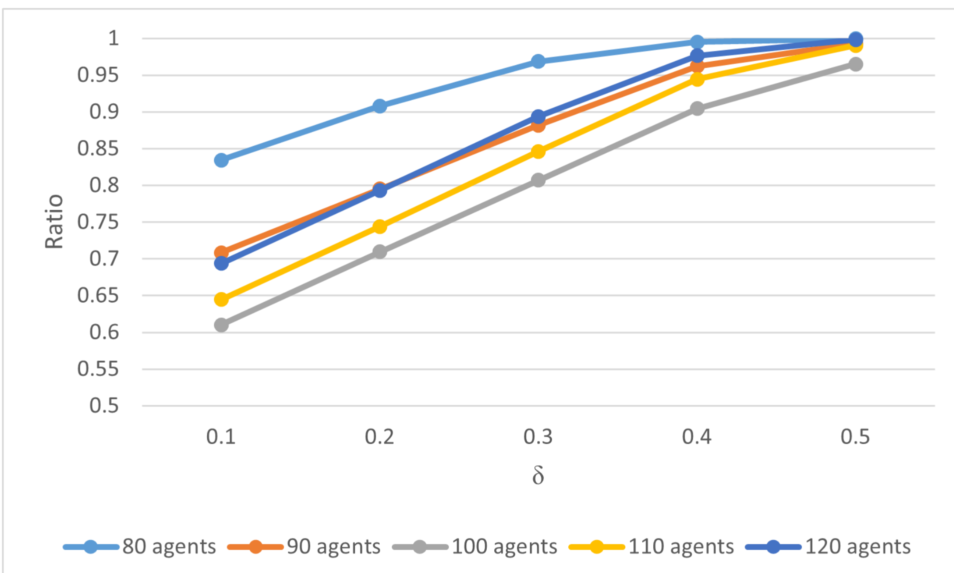


Figure 4: Ratio of seated passengers under MSD and ASD (3-seat restricted case)

We also conduct a simulation analysis under the unrestricted case, i.e., females and males can sit next to each other. Under the unrestricted case, both MSD and ASD do not waste any seats. That is, either all agents are seated or all seats are allocated to some agents. In this unrestricted case, stability implies that no agent would like to swap his/her assignment with another agent with a lower priority. As shown in Section 4.2, ASD is stable. By using our simulations, we calculate the fraction of agents who would like to swap their assignment with a lower priority agent under MSD. We call such a situation as priority violation. The results under 2-seat and 3-seat cases for different levels of α and the number of agents are given in Figures 5 and 6, respectively. As the preferences become more correlated, we observe higher levels of priority violations in all cases. Moreover, we observe higher levels of priority violations when there are fewer agents than seats. This is due to the fact that, when preferences are correlated, agents with lower priority are seated in the unfilled rows since they pick later than the higher priority agents.

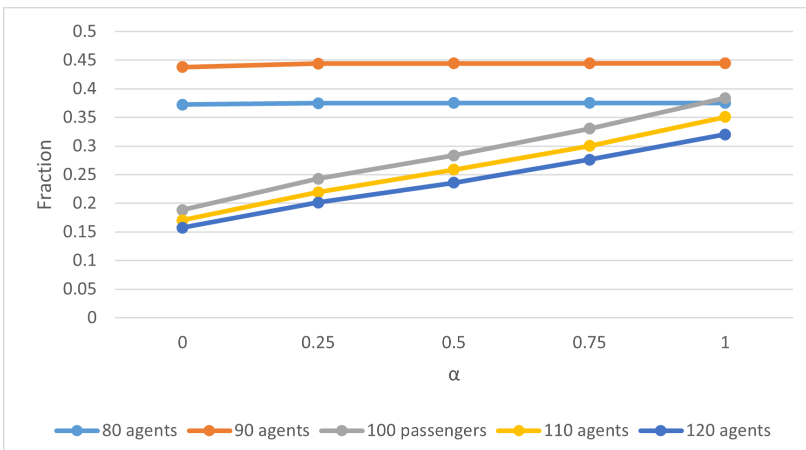


Figure 5: Fraction of agents whose priorities violated under MSD (2-seat unrestricted case)

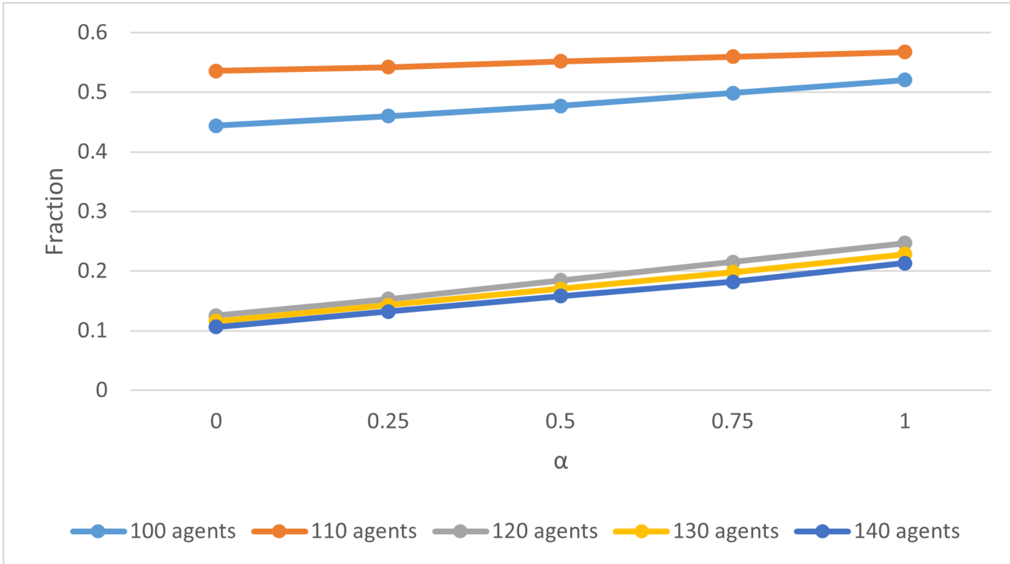


Figure 6: Fraction of agents whose priorities violated under MSD (3-seat unrestricted case)

6 Extensions and Implementation

The benchmark model assumes a specific class of externalities. However, we could consider a more general class of externalities and extend our analysis so long as agents always prefer receiving a seat to being unseated. To formalize this general framework, we let agents have (possibly weak) preferences over matchings. We have the same notions as in the benchmark case. Let us next consider a mechanism where its first stage is the same as ASD's, which finds the set of agents to be seated in the mechanism. In the second stage, we consider the set of all matchings where the only seated agents are those found in the previous step. Then, the first agent in the priority order chooses her best matchings from this set, and all the other matchings are removed. The second agent then chooses her top matching from the reduced set, and so on. This mechanism is stable, efficient, strategy-proof, constrained maximal, and respects improvements.[30] A disadvantage of it is its opacity compared to ASD. This stems from the fact that no specific externality structure is specified, making

[30]Recall that ASD satisfies these properties under the problem studied in this paper.

it impossible to be transparent in its seat assignments. Moreover, calculating the set of all matchings to be considered in the second stage of this mechanism might be computationally cumbersome. For the setting considered in this paper, ASD provides a solution by determining the individuals' assigned seats following the priority ordering and taking the number of individuals to be seated into account, instead of comparing all possible feasible matchings. Hence, we believe that the ASD formulation fits this setting better. Nevertheless, due to the uniqueness result (Theorem 2), ASD and the mechanism mentioned above are outcome-equivalent in our setting.

Another natural extension of our framework is to incorporate couples. Even under gender-based seating restrictions, one may expect that couples can be seated in adjacent seats. Indeed, this is the practice for the Turkish Railways. Fortunately, our algorithm can easily be modified to handle the inclusion of couples in the problem. In the first step, as described above, we treat couples as different agents coming back to back in the ordering (it does not matter who comes first). We elicit one seat ordering from couples. We then apply the same Step 1 of the algorithm except that a couple is not seated whenever the man or the woman in the couple does not receive a seat. Next, they move to the second stage. The second stage works the same with the exception that whenever the woman (the man) in a couple selects a seat, she (he) selects their favorite seat with an empty seat next to it. Otherwise, he/she is assigned as before. In the second stage, if an agent cannot be seated, it is because of the seat allocation of a couple, i.e., the middle seat is given to an agent of a different gender from the former. In this case, we can swap the couple's seats, and the agent can be seated.[31]

We want to discuss to what extent our analysis extends to the weak preferences domain. This is worth touching on as agents' seat rankings may not be strict. We can adapt ASD to this case as follows. We can first obtain a strict ranking over seats by applying a tie-breaking rule. Then, with the obtained strict ordering, we can invoke ASD to find a matching. This matching, however, may not be efficient. To fix this, we can utilize efficiency-improving

[31] A similar extension can be used for families of size three under the case of the 3-seat configuration.

cycles, which have been well-studied in the literature (for instance, see Erdil and Ergin (2008) and Kesten and Ünver (2015)). Cycle construction would take a simple form because the number of seated agents in an agent's row at the ASD matching needs to be fixed. Therefore, we only need to consider agents' seat rankings in constructing cycles. We can define stability preserving and efficiency-improving cycles similar to those that have already been defined and used in the literature. These cycles will also need to take care of feasibility constraints. Such a mechanism would be feasible, efficient, stable, and constrained maximal. However, because of weak rankings, there will be multiple stable matchings, implying that the characterization result (Theorem 2) will no longer hold. Moreover, the mechanism would be manipulable. It may be a fruitful research direction to further study the weak preference domain.

Lastly, we want to elaborate on is the practical implementation of ASD. We statically define the mechanism in the sense that all the agents are pooled, and then it calculates the outcome. However, this is not practical, as agents make reservations over time and need to know whether they can reserve a seat right away. Fortunately, ASD can be implemented dynamically to address this concern. Its first step can be run for each new agent's arrival, and the agent can be informed whether she will be seated. Notice that whether or not an agent is seated only depends on the agents who have arrived earlier. Depending on the trip, this procedure can be conducted subject to a deadline. Then, the further steps determine the final seat assignments. The agents can thus be informed about their seats before the trip. In fact, this implementation is very similar to the seat allocation procedure followed by airlines. Many airline companies sell tickets to passengers without assigning a specific seat at the time of purchase. Passengers are usually informed about their seats when they check-in 24 hours before departure. One might still object ASD's implementation by arguing that eliciting passengers' full-fledged rankings over the seats is impractical. For its remedy, we can limit the information gathered from passengers. For instance, we can only want them to report their preferences over the sides (aisle, window). Of course, in this case, our mechanism outcome will satisfy the properties subject to the partial information. In fact, we can utilize ASD to determine who will be seated next to an empty seat without eliciting the exact preferences of passengers over seats. Then, the passengers who will be seated next to an empty seat will be given a seat holder card to keep their next seat empty. Finally, we can let the passengers select their seats one by one following their priorities (i.e., booking time).

7 Conclusion

Because of harassment, the gender-based seat restriction, which prevents a male-female couple from being seated next to each other, is in effect in intercity transportation in several countries, including Sri Lanka, India, Turkey, and Japan. The first-come-first-served seat allocation results in various problems, including wasted seats and stability violations. This paper introduces a mechanism that can be easily and practically implemented. This mechanism has appealing theoretical properties, including stability, efficiency, and strategy-proofness. Apart from seat allocation in public transportation, it can be applied to other allocation problems where two different types of agents cannot be allocated to two specific objects at the same time. One such problem is hospital bed allocation during a pandemic, where a patient with a contagious illness cannot be assigned to the same room as other patients without a contagious illness.[32]

[32] Here, we can treat each bed in a room to be adjacent to each other and patients with contagious illness as male and the rest as female. As a result, we can modify our proposed mechanisms to determine the room assignment.

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A The General Definitions of ASD

In what follows, we provide the general definition of ASD, covering both two and three-seat cases.

Adaptive Serial Dictatorship (in the Restricted Case)

Step 1. We first tentatively allocate seats to agents. To this end, by following the agent-ordering, we apply the following steps one by one for each agent. For $k \in \{1, .., n\}$,

Substep $1.k$. Let us consider agent i_k. If there is an available seat such that the adjacent seat is taken by an agent of the same gender as i_k while the other adjacent seat (if any) is not taken by an agent of a different gender, then let i_k receive the best one (with respect to $\rhd_{i_k}$) among such seats. Otherwise, if there is an empty row, then let i_k be seated at his/her favorite seat among those in the empty rows. If neither of these holds, the rows contain three seats, and there is an empty seat whose adjacent is not taken, then let i_k receive the best one among such seats.[33] If none of these hold, then let i_k be unseated.

[33]There can already be at most one such seat. However, for the sake of coherence, we let him choose his favorite one.

This procedure terminates by the end of Substep $1.n$. By its termination, if $|\tau_s| = 3$ and there are two rows where one contains only a single man and the other contains a single woman, then we displace the one with a lower priority from her/his seat. We place her/him at the empty non-middle seat in the other agent's row.[34]

Let μ^0 be the tentative matching attained at the end of Step 1. We exclude all the agents in $N \setminus \mu^0_S$ from the problem and they become permanently unassigned. Let us displace the rest from their assignments under μ^0, and each seat becomes available.

Let c be the total number of empty rows under μ^0. If $c > 0$, then we go to Step 2.[35] Otherwise, we go to Step 3.

Step 2. We only consider the top c agents in the ordering. We first let i_1 pick his/her favorite available seat. We remove the agent and the selected row. Among the remaining seats, we repeat the same procedure one by one following the agent-ordering, and it ends after the selection of i_c. We then go to Step 1 in the reduced problem.

Step 3. We have the following exhaustive cases.

Case 1: ***All seats are taken.*** We go to Step 4.

Case 2: ***There is at least one row where only one seat is taken.*** There can be at most two rows where only one seat is taken. We have the following subcases.

Subcase 2.1: ***There are two rows where only one seat is taken.*** This case is possible only if $|\tau_s| = 2$. By Step 1's definition, one of these rows contains a woman, and the other one contains a man. Let us suppose that the top agent under the ordering is a man. The other case follows from the symmetric argument. We let the top agent, who is a man, choose his best seat and remove him along with his row from the problem. We apply Step 4 until the top woman takes her turn. She selects the best remaining seat in a completely empty row. We remove her along with the selected seat and its row from the problem. We then go to Step 4.

[34]Note that since each agent chooses the best seat in completely empty rows whenever s/he is seated at an empty row, the middle-seat in the row is empty. Therefore, the man-woman pair is not seated next to next.

[35]If $c > 0$, then $N = \mu^0_S$.

Subcase 2.2: ***Only one row containing one seated agent whose gender is the same as the top agent.*** We let the top agent choose her/his best remaining seat and remove the selected seat and its row from the problem. If $|\tau_s| = 2$, then we go to Step 4. Otherwise, we go back to Step 1.

Subcase 2.3: ***Only one row containing one seated agent whose gender is different from the top agent.*** Suppose $|\tau_s| = 2$. Let agent j be the top agent of the same gender as the agent whose row has an empty seat. We apply Step 4 till agent j. Whenever it is agent j's turn, we let him/her choose his/her best seat in a completely empty row. We remove the row and go to Step 4.

Suppose $|\tau_s| = 3$. We then calculate the total number of empty seats. If it is equal to 3 (this means that there is another row occupied by two agents of the same gender as the top agent), then let the top ranked agent choose his/her best seat. We remove him/her with the selected seat and its row. We then go back to Step 1.[36]

If it is equal to 2, then we consider the top ranked agent and the two other highest ranked agents of the same gender as the former (the top agent). We also consider the top ranked agent of the other gender. Let us write A for the set of these four agents.

The top agent in A chooses his/her favorite seat, and we remove the seat. The next agent in A selects his/her remaining favorite seat.[37] If these two agents are seated in the same row, then we remove the row. We then apply Step 4 until the third ranked agent in A. Whenever it is her/his turn, s/he chooses her/his best seat in a completely empty row, and the other agent in A is seated at her/his best remaining seat in the selected row. We then remove them along with their selected row and go to Step 4. Otherwise—that is, if the second ranked agent in A is not seated in the same row as the top agent in A—then we let the third agent in A choose his/her favorite remaining seat among the empty seats in the

[36]Notice that, throughout the algorithm, whenever we go back to Step 1, the number of removed agents is always less than the number of removed seats. Hence, we can repeat this procedure finitely many times.

[37]Here, we need Assumption 1 as the top agent in A surely selects a non-middle seat, implying that the other seat in the selected row is available to be chosen by the second agent in A regardless of these agents' genders.

rows taken by the first two agents in A. The last agent in A then chooses his/her favorite remaining seat among these rows. We remove these agents along with their rows and go to Step 4.

Case 3: $|\tau_s| = 3$ *and there is a row with two seated agents of different genders.* By Step 1, there can be at most one such row. In this case, the top agent selects his/her favorite seat. The top agent of the other gender selects the best remaining seat in the selected row by the former. We remove these agents, as well as their selected row, from the problem and go to Step 4.

Case 4: $|\tau_s| = 3$ *and none of the above cases hold.* Let d be the total number of empty seats. Note that by our construction $d \in \{1, 2\}$. We consider the following subcases.

Subcase 4.1: $d = 1$. The row containing the empty seat only includes agents with the same gender (the other case is already addressed in Case 3). Without loss of generality, suppose they are both males. Let j_1 and j_2 be the first and second ranked men. Until j_1's turn, we apply Step 4. Whenever it is his turn, we let j_1 choose his best seat in a completely empty row. We also let j_2 choose his best remaining seat in the row where j_1 is seated. We then remove them along with their row. We then go to Step 4.

Subcase 4.2: $d = 2$. There is one row containing two men and one row containing two women. Let us consider the top two agents from the men and women sides and call this set A. Let the top agent in A select his/her favorite seat and remove it. The next agent in A selects his/her favorite remaining seat. If these two agents are seated in the same row, then we remove the row. We then apply Step 4 until the third-ranked agent in A. Whenever it is her/his turn, we let her/him select the best seat in a completely empty row. Then, the remaining agent in A is seated at her/his best remaining seat in the selected row by the former. We then remove them along with their row and go to Step 4. Otherwise, that is, if the second ranked agent in A is not seated in the same row with the top agent in A, then the third agent in A chooses his/her favorite remaining seat among the empty seats in the rows taken by the first two agents in A. The last agent in A then chooses his/her favorite remaining seat among these rows. We remove these rows and go to Step 4.

Step 4. In the reduced problem, one by one following the agent-ordering, we do the following. We start with the first agent (with respect to the agent-ordering) and let him/her choose his/her best available seat. Let us then consider the second agent. Suppose she is a woman (the other case follows from a symmetric argument). Let us calculate the total number of empty seats in the rows where a woman has already been seated. If this number is equal to the number of unseated women, then we let her choose the best seat in a row where a woman has already been seated. Otherwise,[38] she chooses her best seat in the rows where no man has already been seated. We continue in the same manner until the last agent.

Adaptive Serial Dictatorship (in the Unrestricted Case)

Step 1. We first tentatively allocate seats among agents. To this end, by following the agent-ordering, we apply the following steps one by one for each agent. For $k \in \{1, .., n\}$,

SubStep $1.k$. Let us consider agent i_k. If there is an available seat whose adjacent seat has been already taken, then let i_k receive his/her favorite seat among such seats. Otherwise, if there is an empty row, then let i_k be seated at his/her favorite seat among the ones in the empty rows. If none of these hold, then let i_k be unseated.

This procedure terminates by the end of Substep $1.n$. Let μ^0 be the matching at the end of Step 1. We exclude all the agents in $N \setminus \mu^0_S$ from the problem and let them be permanently unassigned. Let us displace the rest of the agents from their assignments under μ^0, and each seat becomes available to be assigned.

Let c be the total number of empty rows under μ^0. If $c > 0$, then we go to Step 2.[39] Otherwise, we go to Step 3.

Step 2. We only consider the top c agents in the ordering. We first let i_1 pick his/her favorite seat. We remove the agent and the selected row. Among the remaining seats, we repeat the same procedure one by one following the agent-ordering, and it ends after the selection of i_c. We then go to Step 1 in the reduced problem.

[38] This number cannot exceed the number of unseated women.
[39] If $c > 0$, then $N = \mu^0_S$.

Step 3. We have the following exhaustive cases.

Case 1: ***All seats are taken.*** We go to Step 4.

Case 2: ***There is a row with only one seat is taken.*** No other row contains an empty seat. We let the top agent choose her/his best remaining seat. We remove the agent along with the selected row and go to Step 4.

Case 3: $|\boldsymbol{\tau_s}| = \mathbf{3}$ ***and none of the above cases hold.*** There exists only one row with two seated agents while all the others are fully taken. Let j_1 and j_2 be the first and second ranked agents in the reduced problem. We let j_1 select the best remaining seat. Agent j_2 selects the best remaining seat in the row selected by j_1. We then remove them along with the selected row and go to Step 4.

Step 4. In the reduced problem, we let each agent choose his/her best remaining seat one by one following the agent-ordering.

B Proofs

In what follows, all the ASD results' proofs are provided for its general versions defined in Appendix A.

Proof of Proposition 1. Assume for a contradiction that μ is not efficient. Let μ' be a matching such that for each $i \in N$, $\mu' \; R_i \; \mu$, where this relation strictly holds for some agent j. This, as well as the stability of μ, implies that $\mu_S = \mu'_S$.

Let $W = \{i \in N : \; \mu' \; P_i \; \mu\}$. By supposition, $W \neq \emptyset$. Let i_k be the last agent in W (with respect to the agent-ordering). For each $k' < k$, $\mu' \; R_{i_{k'}} \; \mu$, $\mu' \; P_{i_k} \; \mu$, and $\mu'_S = \mu_S$. This, however, contradicts the stability of μ, which finishes the proof. □

Proof of Proposition 2. **MSD is neither stable nor efficient** Let $N = N^m = \{m_1, m_2, m_3, m_4\}$, $S = \{s_1, s_2, s_3, s_4\}$, $\tau_{s_1} = \{s_1, s_2\}$, $\tau_{s_3} = \{s_3, s_4\}$. We will be using the same seats with the same row pattern throughout the proof. Let $m_1 \succ m_2 \succ m_3 \succ m_4$. Agents' strict rankings over S are:

$$s_1 \rhd_{m_1} s_2 \rhd_{m_1} s_3 \rhd_{m_1} s_4$$

$$s_2 \rhd_{m_2} s_1 \rhd_{m_2} s_3 \rhd_{m_2} s_4$$

$$s_3 \rhd_{m_3} s_1 \rhd_{m_3} s_2 \rhd_{m_3} s_4$$

$$s_2 \rhd_{m_4} s_1 \rhd_{m_4} s_3 \rhd_{m_4} s_4$$

In this problem, MSD selects matching μ such that $\mu_{m_1} = s_1$, $\mu_{m_2} = s_3$, $\mu_{m_3} = s_2$, and $\mu_{m_4} = s_4$. Matching μ is not stable, as there exists another matching ν such that $\nu_{m_1} = s_1$, $\nu_{m_2} = s_2$, $\nu_{m_3} = s_3$, and $\nu_{m_4} = s_4$, rendering the violation of stability under μ. Moreover, ν Pareto dominates μ. Therefore, MSD is neither stable nor efficient.

MSD is not maximal: Let $N^m = \{m_1, m_2\}$, $N^f = \{f_1\}$, with $m_1 \succ m_2 \succ f_1$. Agents' strict rankings over S are:

$$s_1 \rhd_{m_1} s_2 \rhd_{m_1} s_3 \rhd_{m_1} s_4$$

$$s_2 \rhd_{m_2} s_1 \rhd_{m_2} s_3 \rhd_{m_2} s_4$$

$$s_3 \rhd_{f_1} s_1 \rhd_{f_1} s_2 \rhd_{f_1} s_4$$

In this problem, MSD selects matching μ where $\mu_{m_1} = s_1$, $\mu_{m_2} = s_3$, and $\mu_{f_1} = \emptyset$. Matching μ is not maximal because there exists another matching ν such that $\nu_{m_1} = s_1$, $\nu_{m_2} = s_2$, and $\nu_{f_1} = s_3$. Hence, MSD is not maximal.

MSD is not strategy-proof: Let $N = N^m = \{m_1, m_2, m_3\}$, with $m_1 \succ m_2 \succ m_3$. Agents' strict rankings over S are:

$$s_1 \rhd_{m_1} s_2 \rhd_{m_1} s_3 \rhd_{m_1} s_4$$

$$s_2 \rhd_{m_2} s_1 \rhd_{m_2} s_3 \rhd_{m_2} s_4$$

$$s_1 \rhd_{m_3} s_2 \rhd_{m_3} s_3 \rhd_{m_3} s_4$$

In this problem, MSD selects matching μ where $\mu_{m_1} = s_1$, $\mu_{m_2} = s_3$, and $\mu_{m_3} = s_2$. However, if agent m_1 reports $s_3 \rhd'_{m_1} s_4 \rhd'_{m_1} s_1 \rhd'_{m_1} s_2$, then MSD selects matching ν such that $\nu_{m_1} = s_3$, $\nu_{m_2} = s_2$, and $\nu_{m_3} = s_1$. Note that $\nu \ P_{m_1} \ \mu$, showing that agent m_1 profitably manipulates MSD. $\square$

Proof of Proposition 3. Consider a problem with $N^m = \{m_1, m_2, m_3, m_4\}$, $N^f = \{f_1\}$, and $S = \{s_1, s_2, s_3, s_4\}$. Let $\tau_{s_1} = \{s_1, s_2\}$ and $\tau_{s_3} = \{s_3, s_4\}$. Let $m_1 \succ m_2 \succ f_1 \succ m_3 \succ m_4$.

Independent of the agent preferences, m_1, m_2, and f_1 have to be seated in any stable matching. However, this implies three seats in total are allocated at any stable matching. On the other hand, we can have all four agents in N^m seated at a matching, showing the incompatibility between stability and maximality. $\square$

The following two lemmas will be critical to the rest of the proofs.

Lemma 1. *Let $\rhd$ be a problem. Once ASD reaches Step 4, in the associated reduced problem, all the agents are seated, and no seat is left empty. In the restricted case, no pair of agents of different gender is seated in the same row in Step 4. Moreover, in both restricted and unrestricted cases, in the associated reduced problem, no agent receives a better outcome without hurting someone else with a higher priority while continuing to assign all the agents.*

Proof. By construction, once ASD reaches Step 4, in the reduced problem, there are just enough empty seats to let the remaining agents receive a seat in a way that no pair of agents of different genders is to be seated in the same row. One by one following the agent-ordering, a woman (man) chooses her (his) favorite empty seat in rows where another woman (man) has been already seated if the total number of empty seats in the rows where a woman (man) has been already seated is just equal to the total number of unassigned women (men). Otherwise, s/he chooses the favorite seat in a row where no agent has been already seated. This construction ensures that no woman (man) selects a seat in a completely empty row that would prevent some man (in the reduced problem) from being unseated due to the gender restriction. As agents choose their favorite empty seats one by one subject to the row selection condition above, no agent can obtain a better outcome without hurting someone else with a higher priority while giving a seat to each agent. $\square$

Lemma 2. *Let $\rhd$ be a problem. Let $ASD(\rhd) = \mu$. Let μ' be another matching where $\mu'_S = \mu_S$. If, for some i_k, $\mu'\ P_{i_k}\ \mu$, then there exists some agent $j \in U(i_k)$ such that $\mu\ P_j\ \mu'$.*

Proof. Let ψ denote ASD. Let μ'' be the outcome of Step 1 of ψ. Since $\mu'_S = \mu_S$, we have $\mu'_S = \mu''_S$. We first claim that the number of empty rows[40] under μ' cannot exceed that under μ''. Let i_k be an agent such that s/he is seated at an empty row in Substep 1.k of ψ. Without loss of generality, let us assume that agent i_k is a man. Note that whenever it is agent i_k's turn, there cannot be a row containing two seated agents of different gender. This is because, by definition, the agent arriving later among these two is to be seated in a completely empty row (we know that there exists such a row as i_k is seated at a completely empty row). Therefore, the only reason why agent i_k is seated at an empty row is that, by his turn, there is no row containing a man and an empty-seat. If $|\tau_s| = 2$, then it directly implies that no row is wasted in ψ, which supports the claim. Let us now assume that $|\tau_s| = 3$. If all the rows including the taken seats by the earlier agents are full, or any row containing an empty seat includes two women, then the only way to give a seat to agent i_k is to assign him a seat in an empty row, implying the claim. Otherwise, there can be a row containing only one woman (note that by our preferential supposition, the woman takes a non-middle seat in the row), say j. However, in Substep 1.k of ψ, instead of placing agent i_k at woman j's row, he is placed at a completely empty row. If there is no other agent after i_k (that is, $|N| = k$), then agent i_k is displaced from his seat and seated at the available non-middle seat at the j's row. Therefore, no row is wasted in the sense that it is not possible to keep the same set of agents seated with fewer rows. Otherwise, if there exists an agent after i_k, i.e., $|N| > k$, then s/he would be seated in an empty row whenever i_k was originally placed at woman j's row. However, as in ψ, whenever i_k is seated at an empty row, the later coming agent is seated either at woman j's row man i_k's row. Therefore, no row is wasted in ψ.

[40] A row is empty if no seat is taken in this row.

This proves the claim.

Let c be the total number of empty rows under μ''. Then, in the course of Step 2, the top c agents choose their favorite seats one by one following the agent-ordering. Moreover, these empty rows are used for the sake of their welfare in that all the other seats in their selected rows are removed from the problem so that each of them is ultimately seated alone. If any of these top c agents receives a different seat under μ', then the agent with the highest priority among them prefers μ to μ'. Therefore, we suppose all top c agents receive the same seat under μ and μ' and they sit alone in the corresponding row. After Step 2, we redo Step 1 in the reduced problem, and so forth. Because of our claim above, in each Step 1 application, the number of empty rows is maximal given the set of seated agents in the reduced problem. All these show that under μ, no agent who is seated alone in a row can be better off without hurting any agent who comes earlier than himself (herself).

Let us now consider Step 3. The algorithm moves to Step 3 whenever there is no empty row left. Note that all the agents who received their assignments and were removed earlier are seated alone in the corresponding rows. If, at the end of Step 1 in the last reduced problem, no seat is left empty (this corresponds to Case 1 of Step 3), then this means that all the agents in the reduced problem are to be seated in full rows. We then go to Step 4. By Lemma 1, each agent in the reduced problem in Step 4 is seated in completely filled rows, and there is no way of making someone among them better off without hurting anyone coming before in the ordering while keeping the same set of assigned agents as under μ.

Let us next consider Case 2 in Step 3. At the end of the last Step 1 application, if $|\tau_s| = 3$, then there can be at most one row containing only one agent, say i_k (by Step 1's definition, it is immediately apparent to see that there cannot be two rows containing only one agent). Let us assume that i_k is a woman. The other case directly follows from symmetric arguments. By the definition of Step 1 of ψ, this implies that no row contains a pair of agents of different genders and, moreover, each row containing a woman other than i_k is full.

We have two cases to consider. First, the top agent, say j, in the reduced problem is a woman (the top agent can be i_k). Note that this case corresponds to Subcase 2.2 in the definition of ASD. This implies that agent j can be seated alone in a row. Hence, we let her choose her best seat and remove her, as well as her selected row, from the problem. We then invoke Step 1 in the reduced problem. Otherwise, agent j, the top agent in the reduced problem, is a man. Note that this case corresponds to Subcase 2.3 in the definition of ASD. We then calculate the total number of empty seats. We already have two empty seats in agent i_k's row. We know that no row containing a woman contains an empty seat. Therefore, if there is an additional empty seat, then it has to come from a row containing two men. By the definition of Step 1, at most one row contains two men, and all the other rows are fully taken. Therefore, the total number of empty seats is either 3 or 2.

Let us assume that it is 3. This means that there is one row containing two men. Recall that we already have a row containing only woman i_k. All the other rows are fully taken. This case implies that the top agent, who is a man, can be seated alone. Therefore, we let him choose his best remaining seat in the reduced problem and remove him, as well as his selected row, from the problem. We then run Step 1 in the reduced problem.

Otherwise, the total number of empty seats is 2. This means that all the rows, except the one containing i_k, are fully taken. In this case, the top agent, who is a man, cannot be seated alone. However, he can be seated in a pair. As the only unfilled row contains only one woman, i_k, and all the other rows are fully taken, it implies that three men and one woman can be seated in pairs, and all the others are to be seated in full rows. Let A be the set with the three top-ranked men and the top-ranked woman (with respect to the agent-ordering). The first top two agents in A choose their favorite available seats one by one, starting with the top-ranked agent. If they are seated in the same row, then we run Step 4 till the third-ranked agent in A, ensuring that all the other agents are to be seated in full rows. Then, the top third agent in A picks his (her) favorite remaining seat, and the remaining agent in A picks his (her) favorite remaining seat in the row selected by the third agent. Otherwise, if the top-two ranked agents in A are seated at different rows, then the third-ranked agent picks his (her) favorite seat among the remaining ones in those two rows, and the fourth agent picks his (her) favorite seat in the remaining row. Hence, these four agents' welfare is maximized as much as possible in the reduced problem. We remove them, along with their selected rows. The algorithm then goes to Step 4. All the remaining agents must be seated in full rows. All these, as well as Lemma 1, show that under μ, no agent can be better off without hurting someone coming earlier than himself (herself) while keeping the same set of assigned agents as under μ.

On the other hand, if $|\tau_s| = 2$, there can be at most two rows, each containing one agent. Let us assume that there are two such rows (this case corresponds to Subcase 2.1 in the definition of ASD). By Step 1's definition, one of these rows contains a man, and the other contains a woman. This implies that all the other rows are full, and hence only one man and one woman can be seated alone. Without loss of generality, assume that the top agent is a man. The algorithm lets the top agent choose his best seat and removes him along with his row. Then, Step 4 is applied till the top woman's turn. Whenever it is her turn, she selects the best remaining seat in an empty row. We then remove her along with the selected row. The algorithm then goes to Step 4. All these, as well as Lemma 1, imply the result.

Suppose there is only one row containing only one agent and $|\tau_s| = 2$. This implies that all the other rows are full. Let us assume that the row contains a man. The other case follows from symmetric arguments. As all the other rows are full, it implies that only one man can be seated alone. If the top agent is a man, then he picks his best seat, and we remove his row and go to Step 4. Otherwise, the top agent is a woman, say agent j. In this case, we apply Step 4 till agent j, and whenever it is her turn, she selects her best seat in a completely empty row. We then remove her selected row and go to Step 4 for the rest of the assignments. This, as well as Lemma 1, implies the result.

Let us now consider Case 3 in Step 3. Suppose there is a man-woman pair who is seated in the same row. This case can only happen for $|\tau_s| = 3$. Note that there cannot be more than one such rows. This implies that all the other rows are completely full. Hence, only one man and one woman can be seated in a pair while all the rest have to be seated in full rows. The algorithm then selects the top woman and the top man. The top agent among these chooses his (her) best seat in the reduced problem, and the other is seated at her (his) favorite seat in the selected row by the former. These two agents along with their row are removed from the problem. As the rest are to be seated in full rows, the algorithm goes to Step 4. This, as well as Lemma 1, implies the result.

Let us consider Case 4. The algorithm reaches this case only when $|\tau_s| = 3$. Suppose there is only one empty seat. This means that one row contains two women (or men), and all the other rows are fully taken (note that the case where a row contains a man and a woman is addressed in Case 3). Suppose that the non-full row contains two women. This case implies that only two women can be seated in a pair, and all the others have to be seated in full rows. In ψ, until the first-ranked woman, we apply Step 4. Whenever it is her turn, she chooses her favorite available seat in a completely empty row. We also let the second-ranked woman choose her best available seat in that row. We then remove them along with their row. We then proceed to Step 4 for the rest. Hence, the two top-ranked women enjoy having an empty seat while all the rest are seated through Step 4. This, as well as Lemma 1, implies the result.

Let us next consider the case where there are two empty seats in total. This implies that there are two rows, each containing one empty seat (note that no row contains a pair of agents of different genders). All the other rows are fully taken. This implies that only two men and two women can be seated in pairs. In ψ, the two top men and two top women are selected. Let us write A for the set of these four agents. The rest of this case is the same as the relevant part in the case above where we consider only one row containing one agent and the total number of empty seats is 2.

By the definition of Step 1, all these cases are exhaustive—that is, there is no other case left. Moreover, in each case where the algorithm goes back to Step 1, some row is removed. Hence, the algorithm ultimately falls into Step 4. Hence, by Lemma 1, we have the result. □

Proof of Theorem 1. Consider an arbitrary problem $\triangleright$. Let $ASD(\triangleright) = \mu$. We first show that μ is stable. Let us pick an agent i_k with $\mu_{i_k} = \emptyset$. Then, by the definition of Step 1 of ASD, it is not feasible to place agent i_k unless some agent in $\mu_S \cap U(i_k)$ loses his seat under μ and is unassigned. Let us now assume that $\mu_{i_k} \neq \emptyset$. By Lemma 2, in order to improve agent i_k's outcome while keeping the same set of assigned agents, some better ranked agent has to be worse off. Therefore, stability is not violated. All these show that μ is stable. By Proposition 1, stability implies Pareto efficiency.

Finally, we show that no agent can benefit from misreporting under ASD. First notice, agents cannot affect the set of seated agents. Moreover, no agent can affect the seat assignment of the earlier agents. All these, as well as Lemma 2, imply that no agent can benefit by misreporting his preferences under ASD, i.e., ASD is strategy-proof. □

Proof of Theorem 2. Consider an arbitrary problem $\triangleright$. Let $ASD(\triangleright) = \mu$. By Theorem 1, μ is stable. Assume for a contradiction that there exists another stable matching μ'.

We first claim that $\mu_S \setminus \mu'_S = \emptyset$. Assume for a contradiction that $\mu_S \setminus \mu'_S \neq \emptyset$. Let i be the best ranked agent in $\mu_S \setminus \mu'_S$ according to $\succ$. This implies that for each $j \in U(i)$, either $j \in \mu_S \cap \mu'_S$ or $\mu_j = \mu'_j = \emptyset$. Note that $\mu'_i = \emptyset$ and $\mu_i \neq \emptyset$. Therefore, we have a matching μ where $\mu_i \neq \emptyset$ and $U(i) \cap \mu'_S \subseteq U(i) \cap \mu_S$. This contradicts the stability of μ'. Hence, we have $\mu_S \subseteq \mu'_S$. This, as well as the stability of μ, implies that $\mu_S = \mu'_S$.

Suppose that $\mu \neq \mu'$. Without loss of generality, let i_k be the highest-priority agent who is not indifferent between matchings, and, without loss of generality, let $\mu \; P_{i_k} \; \mu'$. Then, μ' cannot be stable, because of the violation of the stability's second condition, contradicting the stability of μ'. □

Proof of Proposition 5. First, if each agent receives a seat under ASD, then there is nothing to prove, as it already implies the maximality of the ASD outcome. Suppose that agent i

does not receive a seat, implying that s/he is not seated in Step 1 of ASD. For the rest of the proof, we only consider the Step 1 of ASD.

Let us first assume that $|\tau_s| = 2$. Agent i cannot be seated in Step 1 whenever either all seats are already taken or there is only one seat left whose adjacent seat is taken by an agent, say j, of a different gender. If agent j is the last agent in the ordering of his/her gender, then one seat is left empty. Otherwise, it is taken as well, implying that no seat is left unassigned. These imply that ASD's outcome, if not maximal, assigns one seat less than a maximal matching.

Let us now consider $|\tau_s| = 3$. Agent i cannot be seated in Step 1 only when there is no seat left in his/her turn, or there is an empty seat in a row, but its adjacent seat is already taken by an agent of a different gender. We now claim that there cannot be more than one empty seat at the Step 1 outcome of ASD. Assume for a contradiction that there are two empty seats. We have two cases. We may have two non-full rows where one of them contains two men, and the other one contains two women (all the other rows are full). In this case, agent i would have received a seat. Otherwise, we may have a row containing two empty seats. In this case, again agent i would have received a seat. This, in turn, shows that at most one seat is left empty under ASD, finishing the proof. $\square$

Proof of Theorem 3. Consider an arbitrary problem $(\rhd, \succ)$. Let $\succ'$ be an improvement over $\succ$ for agent i. Let $ASD(\rhd, \succ') = \mu'$ and $ASD(\rhd, \succ) = \mu$. First, if $\mu_i = \emptyset$, then there is nothing to prove. Let us suppose that $\mu_i \neq \emptyset$. Let i be the k^{th} agent in the ordering under $\succ$, that is, $i = i_k$.

By the definition of Step 1 of ASD, it is immediately apparent that $\mu_S = \mu'_S$. Then, by the stability of μ and μ', it must be that $\mu \ R_{i_1} \ \mu'$ and $\mu' \ R_{i_1} \ \mu$. That is, i_1 is indifferent between μ and μ'. The same is true for all agents until agent i_k under $\succ'$.

Assume for a contradiction that $\mu \ P_{i_k} \ \mu'$. Then, μ' cannot be stable, as $\mu_S = \mu'_S$, all the agents having a higher priority than i_k under $\succ'$ are indifferent between μ and μ', and agent i_k prefers μ to μ'. This shows that $\mu' \ R_i \ \mu$, finishing the proof. $\square$

Proof of Proposition 6. An agent is unassigned under MSD only when no empty seat is left. This shows that MSD is maximal. In the proof of Proposition 2, we show the lack of stability, efficiency, and strategy-proofness of MSD in problems where all the agents are male. Therefore, the same examples show that MSD is not stable, efficient, or strategy-proof in the unrestricted case as well. $\square$