A Generalized Time Iteration Method for Solving Dynamic Optimization Problems with Occasionally Binding Constraints*

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Abstract

We introduce a generalized version of Coleman (1990)'s time iteration method (GTI) for solving dynamic optimization problems. We show an application on an irreversible investment model with labor-leisure choice. The GTI algorithm is simple to implement and provides advantages in terms of speed relative to Howard's (Howard (1960)) improvement algorithm.

JEL Classification: C6, C61, C63, C68

KEY WORDS: DSGE models; Occasionally binding constraints, Computational methods; Policy function iteration; Endogenous grid.

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1 Introduction

Most dynamic models in macroeconomics are in the class of nonlinear rational expectations models which are complex and rich in structure, and do not exhibit a closed-form, analytical solution. A solution, if it exists, can be obtained only through numerical techniques. A strand of literature has focused on solution methods that use the first order conditions (FOCs) of the optimization problem, such as Judd (1992), Maliar and Maliar (2005), Maliar et al. (2011) . These methods, however, can still be time consuming or rely on the assumption that the decision rules are smooth, which may limit their applicability. Moreover, there may be problems with convergence, depending on the initial guess as well (see den Haan and Marcet (1990)).

Coleman (1990) and Baxter (1991) suggest methods which explicitly use Euler equations and a grid to approximate the decision rules. This type of an approach is particularly useful as they do not make use of the smoothness of decision rules, and therefore can be useful for models with occasionally-binding constraints, such as investment irreversibility or heterogeneous-agent incomplete markets models in the spirit of Aiyagari (1994). However, speed can be a major concern as these models can become very sophisticated.

We suggest a method that addresses this issue by generalizing the time iteration method of Coleman (1990), who uses policy function iteration on the Euler equation of a simple RBC model. In time iteration, the aim is to solve a fixed-point equation in the form $c = F(c)$ where c is the optimal consumption function and F is derived from the intertemporal Euler equation. This method has been shown to be equivalent to VFI by Coleman (1990) in the RBC model and in a more general model with occasionally-binding constraints by Rendahl (2015).

Our main contribution is to extend Coleman (1990)'s method to solve an RBC model with occasionally-binding constraints and labor-leisure choice. We call this method the generalized time iteration (GTI). Specifically,

- We use Carroll (2006)'s endogenous grid method (EGM). The main idea behind EGM is to consider the future endogenous state variable fixed and the current endogenous state variable unknown–unlike the conventional approach that takes the current state variable as given and solve forward to find the optimal state variable tomorrow.[1]

- We consider a change of variables in the resource constraint to define a new state variable, which we call 'market resources' (i.e. the sum of output and capital after depreciation). This helps solve for current-period capital and labor only once after convergence is achieved, rather than in every iteration, and reduces computation time dramatically by avoiding, again, these extra root-finding procedures. One advantage of using a policy function-based method like ours, is that, such a switch of state variables is feasible when there are multiple choice variables.[2] While a more general applicability of Carroll (2006)'s method has been explored in VFI-based techniques, the literature has not focused much on the advantages of policy function iteration based techniques.[3]

[1]For instance, in an illustration based on the standard stochastic neoclassical growth model, a solution under EGM involves finding decision rules as a function of next period's capital as opposed to the conventional method that defines decision rules over the current period's capital.

[2]This would not be possible under a VFI based method, as shown by, Barillas and Fernandez-Villaverde (2007) who extends Carroll (2006)'s EGM to include labor-leisure choice.

[3]Other papers that enhance Carroll (2006)'s EGM based on a value function iteration method can be listed as follows. Hintermaier and Koeniger (2010) and Fella (2014) suggest related solution methods for models with occasionally binding constraints and a decision on durables and non-durables, and White (2015) considers the theoretical characterization of EGM with multi-dimensional states and controls.

We compare the speed and accuracy gains from GTI to those of the Howard's improvement algorithm (Howard (1960)), i.e. the standard policy function iteration (PFI). PFI is a high threshold to pass in terms of speed and accuracy. It makes use of each new computed policy function by computing the value of using that policy forever, usually taking a smaller number of steps for convergence than the standard VFI (See Ljungquist and Sargent (2012) and Santos and Rust (2003)).

Significant differences arise in the implementation of GTI and PFI, although they are both based on fixed-point iteration. In particular, PFI tends to suffer from the curse of dimensionality. GTI provides speed advantages. Also, unlike Euler equation-based techniques (such as den Haan and Marcet (1990)), convergence is achieved under GTI with an arbitrary initial guess. Our illustration of GTI is based on an irreversible investment model with labor-leisure choice. An additional static choice variable is known to complicate the solution technique significantly, and the GTI method provides a faster solution algorithm.

2 The Generalized Time Iteration Method (GTI)

2.1 A real business cycle model with a constraint on investment

Following Christiano and Fisher (2000) and Guerrieri and Iacovello (2015) we consider a model with a constraint on investment.[4] The expected life-time utility of a representative household is given by

$$U = \mathbb{E}_0 \sum\nolimits_{t=0}^{\infty} \beta^t [u(c_t) + h(1 - l_t)] \tag{1}$$

where $\beta \in (0, 1)$ is the discount factor, c_t is consumption, $l_t \in [0, 1]$ is labor supply (and hence, $1 - l_t$ is defined as leisure). We assume $u' > 0, u'' < 0$ and $h' < 0, h'' < 0$ and both functions are continuously differentiable. GTI works in models only with utility functions that are separable in consumption and leisure**.** The single aggregate output in the economy, y_t is produced using aggregate capital, k_t, and aggregate labor, l_t, according to a constant returns to scale production function, $y_t = e^{z_t} F(k_t, l_t)$. The total factor productivity (TFP), z_t, is governed by an n-state ($n < \infty$) first-order Markov process defined with an $n \times n$ transition probability matrix $\Pi = [\pi_{ij}]$, where $\pi_{ij} = \Pr(z_{t+1} = z_j | z_t = z_i)$. All elements of Π are non-negative and each row sums up to 1. In any given period, the decisions are made after observing the shock z_t.

Capital depreciates at rate $\delta \in [0, 1]$ in each period. Hence the resource constraint for the social planner's problem can be written as

$$c_t + k_{t+1} = e^{z_t} F(k_t, l_t) + (1 - \delta) k_t. \tag{2}$$

We also have the investment constraint

$$k_{t+1} - (1 - \delta) k_t \geq \phi i^{ss} \tag{3}$$

which also encompasses a case with irreversibility under $\phi = 0$. Setting $\phi > 0$, however, ensures that the constraint is occasionally binding under a standard calibration. The planner's problem is to maximize (1)

[4] The method could be applied to solve workhorse macro models in the heterogeneous-agent incomplete markets literature. Indeed, there are earlier applications of time iteration enhanced with EGM in decentralized economies such as Guerrieri and Lorenzoni (2017) and Kabukcuoglu (2017). The former can be considered a Bewley type economy and the latter is closer to two-country Aiyagari (1995) model with progressive taxes, albeit abstracting from elastic labor. The GTI approach provides an advantage in problems that handles a decision for both capital *and* labor. See also Rendahl (2015) with an application of irreversible investment with no labor-leisure choice.

subject to (2) and (3) , the law of motion for TFP shocks, and the initial conditions k_0 and z_0.

Equilibrium Conditions. The solution can be characterized by the Kuhn-Tucker conditions,

$$u'(c_t) - \lambda_t = \beta \mathbb{E}_{z_{t+1}|z_t}[u'(c_{t+1})\,[e^{z_{t+1}}F_k(k_{t+1}, l_{t+1}) + 1 - \delta] - (1-\delta)\lambda_{t+1}] \tag{4}$$

$$u'(c_t)e^{z_t}F_l(k_t, l_t) = h'(1 - l_t) \tag{5}$$

$$c_t + k_{t+1} = e^{z_t}F(k_t, l_t) + (1-\delta)k_t \tag{6}$$

$$k_{t+1} - (1-\delta)\,k_t - \phi i^{ss} \geq 0 \tag{7}$$

$$\lambda_t \geq 0 \tag{8}$$

$$(k_{t+1} - (1-\delta)\,k_t - \phi i^{ss})\lambda_t \geq 0 \tag{9}$$

and the initial conditions, transversality condition and law of motion for TFP shocks. Notice that the multiplier λ_t is state-dependent and hence, the investment constraint is occasionally binding.

2.2 The Generalized Time Iteration (GTI) Method

In this section, we show how to solve this model based on GTI. This approach requires that we use (4)-(9) to obtain the solutions for consumption, c_t^*, labor, l_t^*, capital, k_t^*, and the multiplier, λ_t^*. Hence, GTI follows closely on Coleman (1990)'s time iteration where one considers iteration over policy functions using the optimality conditions.

Furthermore, to enhance speed we use Carroll (2006)'s EGM and a switch of variables into market resources. This means that instead of defining the grid points over k_t and z_t and finding k_{t+1} that satisfies(4)-(9), we define the grid points over k_{t+1} and z_t to find k_t, solving the problem algebraically. But in order to avoid solving for k_t in every iteration, we will redefine variables in terms of market resources, m_t and deal with solving for k_t only in the final step. The resulting k_t values are often off the grid, as the "endogenous gridpoint method" suggests.

We redefine the endogenous state variable in terms of current period market resources

$$m_t = c_t + k_{t+1} \tag{10}$$

and next period market resources,

$$m_{t+1} = e^{z_{t+1}}F\,(k_{t+1}, l_{t+1}) + (1-\delta)\,k_{t+1}. \tag{11}$$

This redefinition enables us to sidestep the burden of using a nonlinear equation solver to find this period's capital and labor in every iteration. Therefore, we can solve for the decision rules for k_t (*and* l_t) only once, in the final step. This idea was introduced by Carroll (2006) in a model without labor-leisure choice, and such a switch of variables could be feasible under VFI. With the introduction of an additional choice variable, however, this type of a switch is feasible under GTI but not VFI.

As we consider the transformation into market resources, the time-invariant decision rules we aim to solve for are $c_t^* = \tilde{g}_c(m_t^*, z_t)$, $l_t^* = \tilde{g}_l(m_t^*, z_t)$, $k_{t+1}^* = \tilde{g}_k(m_t^*, z_t)$ and $\lambda_t^* = \tilde{g}_\lambda(m_t^*, z_t)$ which will eventually help us find the actual functions $c_t^* = g_c(k_t^*, z_t)$, $l_t^* = g_l(k_t^*, z_t)$, $k_{t+1}^* = g_k(k_t^*, z_t)$ and $\lambda_t^* = g_\lambda(k_t^*, z_t)$ using

(4)-(9) .

We start with an initial guess for two functions, $k_{t+2} = \tilde{g}_k(m_{t+1}, z_{t+1}) = g_k(k_{t+1}, z_{t+1})$ and $\lambda_{t+1} = \tilde{g}_\lambda(m_{t+1}, z_{t+1}) = g_\lambda(k_{t+1}, z_{t+1})$. Notice that m_{t+1} takes values as functions of the points over the grid for k_{t+1}, which means we base our guess on the grid points of k_{t+1}.

With the initial guess for $k_{t+2} = g_k(k_{t+1}, z_{t+1})$, we can solve for l_{t+1} using Newton's method combining (5), (6), and (7) as:

$$\max\{k_{t+2}, (1-\delta)k_{t+1} + \phi i^{ss}\} - e^{z_{t+1}} F(k_{t+1}, l_{t+1}) - (1-\delta)k_{t+1} + u'^{-1}\left[\frac{h'(1-l_{t+1})}{e^{z_{t+1}} F_l(k_{t+1}, l_{t+1})}\right] = 0. \tag{12}$$

Notice that since $k_{t+2} = g_k(k_{t+1}, z_{t+1})$, equation (12) is defined over (k_{t+1}, z_{t+1}) gridpoints. Therefore a solution for next period's labor is also defined over values of (k_{t+1}, z_{t+1}), i.e. $l_{t+1} = g_l(k_{t+1}, z_{t+1})$. We can construct (6) in terms of next period's market resources m_{t+1}, and pin down next period's consumption,

$$c_{t+1} = m_{t+1} - \max\{k_{t+2}, (1-\delta)k_{t+1} + \phi i^{ss}\}. \tag{13}$$

Next, we find current period consumption from (4), assuming the investment constraint (7) is slack, therefore $\lambda_t = 0$. As the utility function is separable in consumption and leisure, and using the initial guess for λ_{t+1} and the functions c_{t+1}, l_{t+1} we found earlier, we can solve for c_t directly from the Euler equation,

$$c_t = u'^{-1}[\beta \mathbb{E}_{z_{t+1}|z_t} u'[g_c(k_{t+1}, z_{t+1})]\,[e^{z_{t+1}} F_k(k_{t+1}, g_l(k_{t+1}, z_{t+1})) + 1 - \delta] - (1-\delta)\lambda_{t+1}]. \tag{14}$$

Then, it is easy to compute current period market resources m_t is found from (10).

We then consider the case $\lambda_t \geq 0$ and (7) is binding. Using (4), the policy functions c_{t+1}, l_{t+1}, the initial guess for λ_{t+1} and the grid points for k_{t+1} and z_{t+1}, we obtain

$$\lambda_t = u'(c_{bind}) - \beta \mathbb{E}_{z_{t+1}|z_t}[u'(c_{t+1})\,[e^{z_{t+1}} F_k(k_{t+1}, l_{t+1}) + 1 - \delta] - (1-\delta)\lambda_{t+1}] \tag{15}$$

where c_{bind}, defined over (k_{t+1}, z_t), is the current period consumption when the investment constraint is binding. To find c_{bind}, we first find the current-period capital k_{bind}, that solves

$$k_{t+1} - (1-\delta)\,k_{bind} - \phi i^{ss} = 0. \tag{16}$$

We find c_{bind}, l_{bind} jointly from equations (5) and (6) over values of (k_{t+1}, z_t) Notice that k_{bind}, c_{bind}, and l_{bind} can be found outside of the iterative cycle, with exact solutions over grid points when the constraint (7) is binding.

Now we will update our guess for k_{t+2} and λ_{t+1} without finding k_t or l_t and only using market resources, m_t. The main idea is that we can consider k_{t+1} a time-invariant function of m_t and z_t, i.e. $k_{t+1} = \tilde{g}_k(m_t, z_t)$. This implies that both k_{t+2} and λ_{t+1} can be expressed as a function of next period's states, (m_{t+1}, z_{t+1}). We can then interpolate both k_{t+2} and λ_{t+1} on m_{t+1} using m_t and z_t (using piecewise cubic hermite interpolating polynomial). With the resulting values for k_{t+2} and λ_{t+1},we update our guess until a stopping criterion is satisfied. Once convergence is achieved with a solution, c_t, we can find l_t and k_t jointly from (5) and (6) with Christopher Sim's csolve.m.

In the final step, we need to find the actual policy functions. We find $k_t^* = min(k_t,\, k_{bind})$. Then, we recover c_t^* and l_t^* using values of k_t^*, k_{t+1} and z_t, from (5)-(6) with csolve.m. The actual λ_t^* can be found similarly, using (15) from the final iteration. We describe the algorithms for GTI and PFI in the online appendix.

3 Parameterization and Numerical Findings

We set $\beta = 0.9896$. We let $u(c) = \theta \ln c$, $h(1-l) = (1-\theta)\ln(1-l)$ where $\theta = 0.357$ which produces a steady state labor value of 0.31. We let $F(k,l) = k^{\alpha} l^{1-\alpha}$, where $\alpha = 0.4$. Capital's depreciation rate is $\delta = 0.0196$. Following Guerrieri and Iacovello (2015) we set $\phi = 0.975$, so that the investment constraint is binding about 40% of the time. We consider the TFP shock process $z_t = \rho z_{t-1} + \varepsilon_t$, where $\varepsilon_t \sim N(0, \sigma^2)$, and set $\rho = 0.95$ and $\sigma = 0.007$.We discretize the process into 9 states following Rouwenhorst (1995) approximation.

We follow the standard procedure in the literature (e.g. Judd (1992), Barillas and Fernandez-Villaverde (2007)) to assess the accuracy of our solutions and calculate the normalized intertemporal and intratemporal Euler equation errors (denoted by EE1 and EE2, respectively) implied by the decision rules. We report the maximum and mean Euler equation errors from the simulation of the model for 10,000 periods.

We use MATLAB version R2018a on a PC with Intel(R) Core(TM) i7-7700 CPU 3.60 GHz (3.60 GHz). Our codes are available for the reproduction of the results. We use linearly-spaced grids and consider the performance of both methods under a range of different number of grid points with $k_1 = 0.3k^{ss}$ and $k_M = 1.8k^{ss}$ where k^{ss} is the steady-state level of capital. We use identical grids when comparing the two methods. Baseline calibration yields $k^{ss} = 23.1$ and $i^{ss} = 0.45$.

As shown in Table 1, the main advantage of the GTI algorithm is its speed. The accuracy remains robust across all grid sizes as GTI involves mostly algebraic operations and nonlinear solvers. When the number of nodes is 500, convergence is achieved in 345 iterations and 28.8 CPU seconds. The mean and maximum intertemporal Euler equation errors (EE1) are -3.78 and -3.31, respectively (in log 10 units). The intratemporal errors (EE2) are much smaller, and in PFI they can be considered close to exact solutions. For PFI, other results seem rather mixed. The method produces relatively large max. Euler equation errors as the solution is found on the grid points, which may be too coarse. Accuracy improvements require finer grid points, which comes at higher time costs. In general, GTI dominates PFI mostly in terms of speed.

Table 1: Results

Grid points	CPU time	Mean EE1 error	Max EE1 error	Mean EE2 error	Max EE2 error	Iterations
GTI						
10	2.4s	-3.72	-3.29	-5.40	-4.12	342
500	28.8s	-3.78	-3.31	-10.35	-5.09	345
$1,000$	57.0s	-3.78	-3.38	-11.37	-5.29	345
$2,000$	1m 47.5s	-3.79	-3.29	-12.32	-5.93	345
PFI						
10	.1s	-2.80	-2.53	-14.82	-13.87	2
500	3m 28.5s	-3.59	-1.22	-13.94	-13.34	11
$1,000$	26m 4s	-3.85	-1.43	-14.28	-13.19	11
$2,000$	3h 35m 13.3s	-3.98	-1.40	-14.49	-13.19	11

Note: We report the mean and maximum of absolute Euler equation errors (in log 10 units). Errors are obtained from a stochastic simulation of 10,000 periods.

The GTI algorithm requires more iterations for convergence. However, the total time spent for the solution of the problem shows that each iteration is completed faster compared to the time spent for each iteration in PFI. The speed in GTI can be attributed to (i) the time convention of grid points such that the grids are defined in terms of tomorrow's capital rather than today's capital (ii) the solution of capital and labor for the current period is made only once, and only at the end of the algorithm when convergence is achieved.

4 Conclusion

The presence of multiple control variables in a dynamic programing problem—for example, that is the case with the endogenous labor-leisure choice in the well-known irreversible investment model—may complicate the procedure to find its solution at the expense of computation time and/or accuracy. In this paper, we propose an easy-to-implement solution method which outperforms the powerful benchmark, the standard policy function iteration, in terms of speed.

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