A Generalized Time Iteration Method for Solving Dynamic Optimization Problems with Occasionally Binding Constraints*

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Abstract

We study a generalized version of Coleman (1990)'s time iteration method (GTI) for solving dynamic optimization problems. Our benchmark framework is an irreversible investment model with labor-leisure choice. The GTI algorithm is simple to implement and provides advantages in terms of speed relative to Howard (1960) improvement algorithm. A second application on a heterogeneous agent-incomplete markets model further explores the performance of GTI.

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KEY WORDS: DSGE models; Occasionally binding constraints, Computational methods; Policy function iteration; Endogenous grid.

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1 Introduction

Most dynamic models in macroeconomics are in the class of nonlinear rational expectations models which are complex and rich in structure, and do not exhibit a closed-form, analytical solution. A solution, if it exists, can be obtained only through numerical techniques. A strand of literature has focused on solution methods that use the first order conditions (FOCs) of the optimization problem, such as Judd (1992), Maliar and Maliar (2005), Maliar et al. (2011). These methods, however, can still be time consuming or rely on the assumption that the decision rules are smooth, which may limit their applicability. Moreover, there may be problems with convergence, depending on the initial guess as well (see den Haan and Marcet (1990)).

Coleman (1990) and Baxter (1991) suggest methods which explicitly use Euler equations and a grid to approximate the decision rules. This type of an approach is particularly useful as they do not rely on the assumption of the smoothness of decision rules, and therefore can be useful for models with occasionally-binding constraints, such as investment irreversibility or heterogeneous-agent incomplete markets models in the spirit of Aiyagari (1994). However, speed can be a major concern as these models can become very sophisticated.

We suggest a method that addresses this issue by generalizing the time iteration method of Coleman (1990), which makes use of policy function iteration on the Euler equation of a simple real business cycle (RBC) model. With time iteration, the aim is to solve a fixed-point equation of the form $c = F(c)$, where c is the optimal consumption function and F is derived from the intertemporal Euler equation. This method has been shown to be (theoretically) equivalent to value function iteration (VFI) by Coleman (1990) in an RBC model and in a multidimensional model with occasionally-binding constraints by Rendahl (2015).

Our main contribution is to enhance Coleman (1990)'s method to solve dynamic optimization problems with occasionally-binding constraints. We call this method the generalized time iteration (GTI).[1] In order to explain the implementation more clearly, we start with an RBC model with labor-leisure choice and investment constraints.

In order to solve the problem, first, we use Carroll (2006)'s endogenous grid method (EGM). The main idea behind the EGM is to assume that the future endogenous state variable is fixed and the current endogenous state variable unknown. This is unlike the conventional approach where the current state variable is given and we solve forward to find the optimal state variable tomorrow.[2]

Second, we consider a change of variables in the resource constraint to define a new state variable in this RBC model, which we call 'market resources' (i.e., the sum of output and capital after depreciation). This helps solve for current-period capital and labor, after convergence is achieved, rather than after every iteration. This reduces computation time dramatically by avoiding a number of extra root-finding procedures. One advantage of using a policy function-based method like ours, is that, such a switch of state variables is feasible when there are multiple choice variables.[3] While a more general applicability of Carroll (2006)'s method has been explored in VFI-based techniques, the literature has not focused much on the advantages of policy function iteration (PFI) based techniques.[4]

[1]The MATLAB programs are available here: http://aysekabukcuoglu.weebly.com/uploads/1/0/1/8/10189079/gti_codes.zip

[2]For instance, in an illustration based on the standard stochastic neoclassical growth model, a solution under EGM involves finding decision rules as a function of next period's capital as opposed to the conventional method that defines decision rules over the current period's capital.

[3]This would not be possible under a VFI based method, as shown by, Barillas and Fernandez-Villaverde (2007) who extends Carroll (2006)'s EGM to include labor-leisure choice.

[4]Other papers that enhance Carroll (2006)'s EGM based on a value function iteration method can be listed as follows. Hintermaier

We compare the speed and accuracy gains from GTI to those of the standard improvement algorithm as discussed in Howard (1960). The standard PFI is a high threshold to pass in terms of both speed and accuracy. It makes use of each new computed policy function after considering the value of using that policy forever, usually taking a smaller number of steps for convergence than the standard VFI (See Ljungquist and Sargent (2012) and Santos and Rust (2003)).

While we obtain comparable results on accuracy, significant differences arise in the implementation of GTI and PFI, although they are both based on fixed-point iteration. In particular, PFI tends to suffer from the curse of dimensionality, where GTI provides notable speed advantages. Also, unlike Euler equation-based techniques (such as den Haan and Marcet (1990)), convergence is achieved under GTI without an educated guess for the policy function of future capital holdings.[5] Our illustration of GTI is based on an irreversible investment model with labor-leisure choice, where the additional static choice variable produces more complex solutions and the benefits of the faster GTI solution algorithm are more apparent.

Other useful applications include the potential use of this solution method in workhorse heterogeneous-agent macroeconomic models that allow for incomplete markets. As a more general application, we include a two-country heterogeneous agent-incomplete markets model in the spirit of Aiyagari (1994) and with progressive labor income taxes. This model extends the one in Kabukcuoglu (2017) by including a labor-leisure choice. The speed and accuracy results of GTI can be seen further in this exercise. Earlier and relatively simple applications of time iteration, enhanced with EGM in decentralized economies were studied by Guerrieri and Lorenzoni (2017) and Kabukcuoglu (2017)[6]

Our final comment pertains to the comparative performance of this method relating to Carroll (2006)'s work that focuses on VFI-based methods. When applying his approach, Carroll (2006) incorporates liquidity constraints in the absence of a labor-leisure choice, while Barillas and Fernandez-Villaverde (2007) generalize Carroll (2006)'s endogenous grid point method in an RBC model with a labor-leisure choice (albeit with no inequality constraints). Barillas and Fernandez-Villaverde (2007) then show the performance of the generalized EGM against the standard VFI, documenting its advantages in terms of speed. Although the VFI is a natural benchmark in their case, it is not a very difficult benchmark to outperform in terms of speed as the curse of dimensionality is an even more serious problem when applying these methods. The current work, therefore, also contributes to the literature by providing a comparison between two powerful methods, and documenting the advantages of time iteration, particularly in terms of speed, in the presence of multiple choice variables and occasionally-binding constraints—an issue which has not been addressed by Barillas and Fernandez-Villaverde (2007).

1.1 Related Literature

There is a vast literature on both local and global solution methods, including perturbation, projection, and value function iteration methods (see, e.g., the literature survey of Fernández-Villaverde et al. (2016)). A model solution can be obtained with global methods but can also be approximated with perturbation methods (first-order, second-order, or even higher-order perturbations). Martínez-García (2018) provides a

and Koeniger (2010) and Fella (2014) suggest related solution methods for models with occasionally binding constraints and a decision on durables and non-durables, and White (2015) considers the theoretical characterization of EGM with multi-dimensional states and controls.

[5] Specifically, this is the policy function for k'' in our model. Convergence is not guaranteed when applying den Haan and Marcet (1990)'s parameterized expectations approach, even when making use of an educated guess.

[6] See also Rendahl (2015) with an application of irreversible investment with no labor-leisure choice.

general introduction to the first-order perturbation method, while Schmitt-Grohé and Uribe (2004) shows how to solve the model using a second-order approximation to the policy function. Andreasen et al. (2018) discuss second- and higher-order approximations and establish the stability of the economic system with a pruned state-space system. More directly related to our paper, the perturbation method can also be used in the presence of occasionally-binding constraints with the technique of Guerrieri and Iacoviello (2015) (in short, OccBin). Holden (2016) and Holden (2019) develop a more general procedure that improves on Occbin by having some guaranteed convergence properties (if a solution exists) and some extensions for higher-order perturbation.

The key advantage of perturbation methods is its tractability and flexibility when the economic system has a large number of state variables, as shown by Aruoba et al. (2006). However, perturbation methods are local approximations and, therefore, are better suited to approximate the policy function when focusing on small perturbations around the deterministic steady state. Needless to say, the approximation is less reliable when shocks are large or when the economy appears to be far from the steady-state if the equilibrium conditions of the model are highly nonlinear. A second- or higher-order perturbation method can help approximate the nonlinear features of the policy function with a polynomial (cf., Balke et al. (2017)). Even so, higher-order approximations generally have difficulties handling problems in which policy function has kinks such as those that arise from occasionally-biding constraints.

Furthermore, the available class of perturbation-based methods depends in general on the existence of a deterministic steady state in which the constraint always binds which is not generally a property of the model. Furthermore, for the OccBin method, being at the occasionally-binding constraint is like imposing an MIT-type shock because the economy gets there completely unexpectedly and agents act under the expectation that they will not return to the constraint again in the future. All of these limitations of the different perturbation-based approaches and of popular methods like OccBin means that using them imposes some strong constraints on the features of the model under which the policy function can be consistently recovered. The approach proposed in this paper has the advantage that it provides a global solution method that overcome the major limitations of the perturbations method noted here.

Occasionally-binding constraints can be better-handled with projection methods. For this, the approach consists in projecting the policy function of the model onto some basis functions (Judd (1992)). The basis can be chosen globally (being nonzero and smooth for most of the domain, i.e., the spectral method) or locally (being zero for most of the domain, i.e., the finite-element method). One commonly used basis function is the Chebyshev polynomial, but linear splines can often be a convenient and reliable choice with which to approximate policy functions that have kinks such as it is often the case in the presence of occasionally-binding constraints. Christiano and Fisher (2000) advocate instead the use of an adapted version of the parameterized expectations approach (PEA) such as in den Haan and Marcet (1990) which appear to dominate the projection methods (spectral methods but also some finite element methods) on the basis of speed and accuracy even though convergence cannot be guaranteed when using PEA. Furthermore, the projection methods suffer from the curse of dimensionality. For that reason, Malin et al. (2011) among others would advocate the use of the Smolyak collocation method in order to simplify the computational burden of projections in multidimensional cases.

GTI allows for a flexible computation of the policy function with and without kinks. We show in the remainder of the paper that this strategy is efficient in terms of time and computational resources. In particular, it maintains accuracy in models with and without kinks introduced by occasionally-binding constraints when compared to conventional PFI.

The method we study in this paper is related to the well-known value function iteration method and most closely to the time-iteration method. These methods are popular because they can deal with kinks in policy function as well as with rich economic models (e.g. heterogeneous-agent models) and models where a deterministic steady-state may not exist. However, the standard applications of these methods are also subject to the curse of dimensionality and the choice of grids, in particular, is crucial to determine the computational complexity involved in solving the model. There are several popular methods for choosing the grid points that simplify the computation and can improve the accuracy in recovering the policy function: the quadrature method by Tauchen and Hussey (1991), the randomized grid method by Rust (1997), and the EGM of Carroll (2006) generalized by Barillas and Fernandez-Villaverde (2007) to capture labor-leisure choice.

More recently, the envelope condition method (ECM) by Maliar and Maliar (2013) utilizes the envelope condition as opposed to the first order conditions that are used in standard VFI or EGM. This approach helps reduce the cost of standard VFI. The application of ECM in an RBC model with labor-leisure attains high speed and accuracy results that are comparable to the EGM by Carroll (2006).[7] In light of this, we did consider experiments with a variant of ECM that iterates over policy functions—a technique which can be compared to GTI.[8] Accordingly, we find the following results. First, ECM yields highly desirable accuracy and speed results in a model with smooth policy functions. Moreover, higher order polynomials used in policy and value function approximation enhance both the speed and accuracy of ECM. Second, in a model with kinks, accuracy of ECM deteriorates as one needs to use different approximation methods that handle kinks better. We consider this an interesting issue to be addressed in future work.

Our key contribution in this paper is to combine time iteration with the EGM in the presence of the kinks in the policy function that arise from occasionally-binding constraints. Hence a more efficient algorithm allows us to speed up the computation of the solution and the recovery of the policy function, while its accuracy remains comparable to that of conventional PFI.

In the next section, we introduce the GTI method, and describe the algorithm for the solution of an RBC model with investment irreversibility. In section 3, we explain the solution under policy function iteration and list the steps of the algorithm. In section 6, we present an application of GTI in a heterogeneous agent-incomplete markets framework. In section 7, we conclude.

2 The Generalized Time Iteration Method (GTI)

2.1 A Real Business Cycle Model with a Constraint on Investment

Following Christiano and Fisher (2000) and Guerrieri and Iacoviello (2015) we consider a model with a constraint on investment. The expected life-time utility of a representative household is given by

$$U = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t [u\,(c_t) + h(1 - l_t)] \tag{1}$$

[7]The mathematical properties and different applications of ECM, including a method that is based on policy function iteration, are studied in Arellano et al. (2016).

[8]These results are not presented in the current work and are available upon request.

where $\beta \in (0,1)$ is the discount factor, c_t is consumption, $l_t \in [0,1]$ is labor supply (and hence, $1 - l_t$ is defined as leisure). We assume $u' > 0, u'' < 0$ and $h' < 0$, $h'' < 0$ and both functions are continuously differentiable. GTI can only be applied in models with utility functions that are separable in consumption and leisure**.** The single aggregate output in the economy, y_t is produced using aggregate capital, k_t, and aggregate labor, l_t, according to a constant returns to scale production function, $y_t = e^{z_t} F(k_t, l_t)$. The total factor productivity (TFP), z_t, is governed by an n-state ($n < \infty$) first-order Markov process defined with an $n \times n$ transition probability matrix, $\Pi = [\pi_{ij}]$, where $\pi_{ij} = \Pr(z_{t+1} = z_j | z_t = z_i)$. All elements of Π are non-negative and each row sums up to 1. In any given period, the decisions are made after observing the shock z_t.

Capital depreciates at rate $\delta \in [0,1]$ in each period. Hence the resource constraint for the social planner's problem can be written as

$$c_t + k_{t+1} = e^{z_t} F(k_t, l_t) + (1-\delta) k_t. \tag{2}$$

We also have the investment constraint

$$k_{t+1} - (1-\delta) k_t \geq \phi i^{ss} \tag{3}$$

which encompasses a case with irreversibility under $\phi = 0$. Setting $\phi > 0$, however, ensures that the constraint is occasionally binding under a standard calibration. The planner's problem is to maximize (1) subject to (2) and (3) , the law of motion for TFP shocks, and the initial conditions k_0 and z_0.

Equilibrium Conditions. The solution can be characterized by the Kuhn-Tucker conditions,

$$u'(c_t) - \lambda_t = \beta \mathbb{E}_{z_{t+1}|z_t}[u'(c_{t+1}) [e^{z_{t+1}} F_k(k_{t+1}, l_{t+1}) + 1 - \delta] - (1-\delta)\lambda_{t+1}] \tag{4}$$

$$u'(c_t) e^{z_t} F_l(k_t, l_t) = h'(1 - l_t) \tag{5}$$

$$c_t + k_{t+1} = e^{z_t} F(k_t, l_t) + (1-\delta) k_t \tag{6}$$

$$k_{t+1} - (1-\delta) k_t - \phi i^{ss} \geq 0 \tag{7}$$

$$\lambda_t \geq 0 \tag{8}$$

$$(k_{t+1} - (1-\delta) k_t - \phi i^{ss}) \lambda_t \geq 0 \tag{9}$$

and the initial conditions, transversality condition and law of motion for TFP shocks. Notice that the multiplier λ_t is state-dependent and hence, the investment constraint is occasionally binding.

2.2 The Generalized Time Iteration (GTI) Method

In this section, we show how to solve this model based on GTI. This approach requires that we use (4)-(9) to obtain the solutions for consumption, c_t^*, labor, l_t^*, capital, k_t^*, and the multiplier, λ_t^*. Hence, GTI follows closely on Coleman (1990)'s time iteration where one considers iteration over policy functions using the optimality conditions.

Furthermore, to enhance speed we use Carroll (2006)'s EGM and a switch of variables into market resources. This means that instead of defining the grid points over k_t and z_t and finding k_{t+1} that satisfies (4)-(9), we define the grid points over k_{t+1} and z_t to find k_t, solving the problem algebraically. But in order to avoid solving for k_t during every iteration, we redefine variables in terms of market resources, m_t and deal with solving for k_t only in the final step. The resulting k_t values are often off the grid, as the "endogenous gridpoint method" suggests.

We redefine the endogenous state variable in terms of current period market resources

$$m_t = c_t + k_{t+1} \tag{10}$$

and next period market resources,

$$m_{t+1} = e^{z_{t+1}} F\left(k_{t+1}, l_{t+1}\right) + (1-\delta)\, k_{t+1}. \tag{11}$$

This redefinition enables us to sidestep the burden of using a nonlinear equation solver to find this period's capital and labor in every iteration. Therefore, we only solve for the decision rules for k_t (and l_t) once, in the final step. This idea was introduced by Carroll (2006) in a model without labor-leisure choice, and such a switch of variables could be feasible under VFI. With the introduction of an additional choice variable, however, this type of a switch is feasible under GTI but not VFI.

As we consider the transformation into market resources, the time-invariant decision rules we aim to solve for are $c_t^* = \tilde{g}_c(m_t^*, z_t)$, $l_t^* = \tilde{g}_l(m_t^*, z_t)$, $k_{t+1}^* = \tilde{g}_k(m_t^*, z_t)$ and $\lambda_t^* = \tilde{g}_\lambda(m_t^*, z_t)$ which will eventually help us find the actual functions $c_t^* = g_c(k_t^*, z_t)$, $l_t^* = g_l(k_t^*, z_t)$, $k_{t+1}^* = g_k(k_t^*, z_t)$ and $\lambda_t^* = g_\lambda(k_t^*, z_t)$ using (4)-(9) .

We start with an initial guess for two functions, $k_{t+2} = \tilde{g}_k(m_{t+1}, z_{t+1}) = g_k(k_{t+1}, z_{t+1})$ and $\lambda_{t+1} = \tilde{g}_\lambda(m_{t+1}, z_{t+1}) = g_\lambda(k_{t+1}, z_{t+1})$. Notice that m_{t+1} takes values as functions of the points over the grid for k_{t+1}, which implies that we base our guess on the grid points of k_{t+1}.

With the initial guess for $k_{t+2} = g_k(k_{t+1}, z_{t+1})$, we can solve for l_{t+1} using Newton's method combining (5), (6), and (7) as:

$$\max\{k_{t+2}, (1-\delta)k_{t+1} + \phi i^{ss}\} - e^{z_{t+1}} F\left(k_{t+1}, l_{t+1}\right) - (1-\delta)k_{t+1} + u^{'-1}\left[\frac{h'\left(1-l_{t+1}\right)}{e^{z_{t+1}} F_l(k_{t+1}, l_{t+1})}\right] = 0. \tag{12}$$

Notice that since $k_{t+2} = g_k(k_{t+1}, z_{t+1})$, equation (12) is defined over (k_{t+1}, z_{t+1}) gridpoints. Therefore a solution for next period's labor is also defined over values of (k_{t+1}, z_{t+1}), i.e. $l_{t+1} = g_l(k_{t+1}, z_{t+1})$. We can construct (6) in terms of next period's market resources m_{t+1}, and pin down next period's consumption,

$$c_{t+1} = m_{t+1} - \max\{k_{t+2}, (1-\delta)k_{t+1} + \phi i^{ss}\}. \tag{13}$$

Next, we find current period consumption from (4), assuming the investment constraint (7) is slack, therefore $\lambda_t = 0$. As the utility function is separable in consumption and leisure, and using the initial guess for λ_{t+1} and the functions c_{t+1}, l_{t+1} we found earlier, we can solve for c_t directly from the Euler equation,

$$c_t = u'^{-1}[\beta \mathbb{E}_{z_{t+1}|z_t} u'[g_c(k_{t+1}, z_{t+1})]\,[e^{z_{t+1}} F_k(k_{t+1}, g_l(k_{t+1}, z_{t+1})) + 1 - \delta] - (1-\delta)\lambda_{t+1}]. \tag{14}$$

Then, it is easy to compute current period market resources m_t is found from (10).

We then consider the case $\lambda_t \geq 0$ and (7) is binding. Using (4), the policy functions c_{t+1}, l_{t+1}, the initial guess for λ_{t+1} and the grid points for k_{t+1} and z_{t+1}, we obtain

$$\lambda_t = u'(c_{bind}) - \beta \mathbb{E}_{z_{t+1}|z_t}[u'(c_{t+1})\,[e^{z_{t+1}}F_k(k_{t+1}, l_{t+1}) + 1 - \delta] - (1-\delta)\lambda_{t+1}], \tag{15}$$

where c_{bind}, defined over (k_{t+1}, z_t), is the current period consumption when the investment constraint is binding. To find c_{bind}, we first find the current-period capital k_{bind}, that solves

$$k_{t+1} - (1-\delta)\,k_{bind} - \phi i^{ss} = 0. \tag{16}$$

We are then able to find c_{bind}, l_{bind} jointly from equations (5) and (6) over values of (k_{t+1}, z_t). Notice that k_{bind}, c_{bind}, and l_{bind} can be found outside of the iterative cycle, with exact solutions over grid points when constraint (7) is binding.

Our guess for k_{t+2} and λ_{t+1} is then updated without finding k_t or l_t and only using market resources, m_t. The main idea is that we can consider k_{t+1} a time-invariant function of m_t and z_t, i.e. $k_{t+1} = \tilde{g}_k(m_t, z_t)$. This implies that both k_{t+2} and λ_{t+1} can be expressed as a function of next period's states, (m_{t+1}, z_{t+1}). We can then interpolate both k_{t+2} and λ_{t+1} on m_{t+1} using m_t and z_t (using piecewise cubic hermite interpolating polynomials). With the resulting values for k_{t+2} and λ_{t+1}, we update our guess until a stopping criterion is satisfied. Once convergence is achieved with a solution, c_t, we can find l_t and k_t jointly from (5) and (6) with Christopher Sim's csolve.m routine.

In the final step, we need to find the actual policy functions. We find $k_t^* = min(k_t, k_{bind})$. Then, we recover c_t^* and l_t^* using values of k_t^*, k_{t+1} and z_t, from $(5) - (6)$ where we make use of Christopher Sim's csolve.m routine once again. The actual λ_t^* can be found similarly, using (15) from the final iteration. Below we describe the GTI algorithm in further detail.

2.3 The GTI Algorithm

1. Find the exact solutions for k_{bind} from (7). Find c_{bind} and l_{bind} jointly from (5) and (6) over values of $(k_{t+1}, z_t) \in \mathcal{G}_{k_{t+1}} \times \mathcal{G}_{z_t}$ using Christopher Sim's csolve.m.

2. Set $i = 0$ and make a policy function guess $k_{t+2}^i = \tilde{g}_k^i(m_{t+1}, z_{t+1}) = g_k^i(k_{t+1}, z_{t+1})$. We start with a guess that sets $k_{t+2}^i = (k^{ss})^\alpha (l^{ss})^{1-\alpha}$, for all state pairs $(k_{t+1}, z_{t+1}) \in \mathcal{G}_{k_{t+1}} \times \mathcal{G}_{z_t}$. Make a guess for the multiplier such that $\lambda_{t+1}^i = \tilde{g}_\lambda^i(m_{t+1}, z_{t+1}) = g_\lambda^i(k_{t+1}, z_{t+1}) = 0$ for all $(k_{t+1}, z_{t+1}) \in \mathcal{G}_{k_{t+1}} \times \mathcal{G}_{z_t}$and initialize the current period market resources $m_t^i(k_{t+1}, z_t) = 0$ for all $(k_{t+1}, z_t) \in \mathcal{G}_{k_{t+1}} \times \mathcal{G}_{z_t}$.

3. For each point of k_{t+2}^i and associated state $(k_{t+1}, z_{t+1}) \in \mathcal{G}_{k_{t+1}} \times \mathcal{G}_{z_t}$, solve the nonlinear equation for l_{t+1} and c_{t+1} using Newton's method jointly from (5)-(7) which is quite fast.[9] Then find c_t from (4) over the grid points (k_{t+1}, z_t) using (4) for the case $\lambda_t = 0$. Using these decision rules, compute the current and next period's market resources m_t^{i+1} and m_{t+1}^{i+1}, respectively. Then compute λ_t treating it as a residual in (4), with $c_t = c_{bind}$.

4. Check if $\sup_{m,n} |m_t^{i+1}(k_m, z_n) - m_t^i(k_m, z_n)| \geq 1.0e^{-6}$. If convergence is not achieved, let $i \rightsquigarrow i+1$ and $m_t^{i+1} = m_t^i$. Update the decision rule for capital using interpolation. In particular, we use piecewise

[9]Other alternatives for MATLAB such as fsolve and csolve (by Christopher Sims) appear to yield the same results but require more computation time.

cubic hermite interpolating polynomials ('pchip' in MATLAB) to interpolate $\tilde{g}_k^{i+1}(m_{t+1}, z_{t+1})$ on m_{t+1}^{i+1} using m_t^{i+1}.[10] Do another interpolation step for $\tilde{g}_\lambda^i(m_{t+1}, z_{t+1})$. Go to step 2.

5. If convergence is achieved, using c_t, find l_t and k_t jointly from (5) and (6) using a nonlinear equation solver. Here, we use Christopher Sim's csolve.m function in MATLAB. The resulting values from this step can be used to find the actual policy function $k_t^* = min(k_t, k_{bind})$ given (k_{t+1}, z_t). The states k_t^*, k_{t+1} and z_t enable us to find actual values for c_t^* and l_t^* from equations (5) and (6). Hence we find the solution to the problem, $c_t = g_c(k_t^*, z_t)$, $l_t = g_l(k_t^*, z_t)$ and $k_{t+1} = g_k(k_t^*, z_t)$.

3 Policy Function Iteration (PFI)

The standard PFI is known to be a very powerful method for solving this class of models (see, e.g., Ljungquist and Sargent (2012), pp. $106 - 107$) as it provides convergence at a quadratic rate, rather than a linear rate, as in the case of value function iteration (Puterman and Brumelle (1979) and Santos and Rust (2003)). It is also a natural benchmark for us since GTI relies on policy function iteration. In this section we will describe how the current model with a labor-leisure choice can be solved with the PFI method. In order to iterate on a policy function, we need to express the problem in recursive form.

Defining k_{t+1} as our control variable and treating the intratemporal FOC (5) as an additional constraint, we plug in for l_t and c_t using (5) and (6), in order to express the period utility function for all values of z_t, k_t and k_{t+1}. Hence we define the dynamic programming problem as follows:

$$V(k_t, z_t) = \max_{\substack{k_{t+1} \\ s.t.\ k_{t+1} \geq (1-\delta)k_t + \phi i^{ss}}} \left\{ u(z_t, k_t, k_{t+1}) + h(z_t, k_t, k_{t+1}) + \beta \mathbb{E}_{z_{t+1}|z_t} V(k_{t+1}, z_{t+1}) \right\}.$$

This problem can also be solved by defining the control variables as k_{t+1} and l_t and using additional grids for labor. However, the method performs poorly in terms of speed and accuracy and becomes a weaker benchmark to compare against GTI.

Then we pick a feasible policy function, $k_{t+1} = g_k^i(k_t, z_t)$ and compute the value associated with the infinite horizon problem using this policy,

$$V_i(g_k^i(k_t, z_t), z_{t+1}) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(k_t, g_k^i(k_t, z_t)), \tag{17}$$

with $i = 0$. Next, we consider the policy improvement step, where the policy function solves

$$\tilde{g}_k^{i+1}(k_t, z_t) = \arg\max_{k_{t+1}} \left\{ u(z_t, k_t, k_{t+1}) + h(z_t, k_t, k_{t+1}) + \beta \mathbb{E}_{z_{t+1}|z_t} V_i(g_k^i(k_t, z_t), z_{t+1}) \right\} \tag{18}$$

to find the actual policy function,

$$g_k^{i+1}(k_t, z_t) = \max[\tilde{g}_k^{i+1}(k_t, z_t), (1-\delta)k_t + \phi i^{ss}], \tag{19}$$

and the associated value function. Notice that this policy function $g_k^{i+1}(k_t, z_t)$ will lie off the grids when the constraint is binding, so we pick the nearest gridpoint k_{t+1} for the resulting policy function. Then iterate

[10] Different interpolation techniques are studied in detail in Judd (1998) pp. 216-235.

over i until convergence is obtained for steps (17) and (18).

3.1 The PFI Algorithm:

We define the grid points for capital, $\mathcal{G}_k \equiv \{k_1, k_2, ..., k_M\}$, and use the Rouwenhorst (1995) approximation method to obtain the discretized stochastic process for the total factor productivity shocks, defined with grid points $\mathcal{G}_{z_t} \equiv \{z_1, z_2, ..., z_N\}$ with the associated transition probability matrix Π. The steps of the algorithm are described as follows:

1. For each triplet of productivity shock, today's and tomorrow's capital $(z_t, k_t, k_{t+1}) \in \mathcal{G}_k \times \mathcal{G}_k \times \mathcal{G}_z$, we construct the matrix $u(z_t, k_t, k_{t+1}) + h(z_t, k_t, k_{t+1})$. In order to do so, we solve the nonlinear equation resulting from (5) and (6) for l_t using Newton's method and find c_t using (6), over the grid points (k_t, z_t).[11]

2. Set $i = 0$ and construct the initial value function $V_i(k_t, z_t)$. In order for this to be consistent with the initial guess in GTI, we set $V_i(k_t, z_t) = u(c^{ss}) + h(1 - l^{ss})$ for all $(z_t, k_t, k_{t+1}) \in \mathcal{G}_k \times \mathcal{G}_k \times \mathcal{G}_z$.

3. We find the decision rule $g_k^{i+1}(k_t, z_t) = \arg\max\{u(z_t, k_t, k_{t+1}) + h(z_t, k_t, k_{t+1}) + \beta \mathbb{E}_{z_{t+1}|z_t} V_i(g_k^i(k_t, z_t), z_{t+1})\}$.

4. We then need to compute the value of using this policy forever and solve forward the Bellman equation in (18) to find the new value function V_{i+1}.

5. Check if $\sup_{m,n} |V_{i+1}(k_m, z_n) - V_i(k_m, z_n)| \geq 1.0e^{-6}$. If convergence is not achieved, go to step 2 and let $i \rightsquigarrow i + 1$.

6. If convergence is achieved, find $l_t = g_l(k_t, z_t)$ using Newton's method and $c_t = g_c(k_t, z_t)$ from the resource constraint.

Even though the construction of $u(z_t, k_t, k_{t+1})$ in step 1 of PFI is done only once, it requires the use of a numerical solver $N_k \times N_k \times N_z$ times. In GTI however, this procedure is repeated $N_k \times N_z$ times, for each iteration when obtaining the labor decision rules in step 2. It requires a quantitative exercise to find out which method is more-time consuming in this step. Our numerical experiments reveal that we need a sufficiently large N_k to obtain more accurate results with PFI, and in this case it is also slower than GTI. The rest of the speed advantages in GTI can be attributed to the time-convention in the grid points and the redefinition of the state variable in terms of market resources.

4 Parameterization and Numerical Findings

Let the intertemporal discount factor be $\beta = 0.9896$ and the instantaneous utility function be given by $u(c) = \theta \ln c$, $h(1 - l) = (1 - \theta) \ln(1 - l)$ where $\theta = 0.357$. This produces a steady state value for labor of 0.31. We let $F(k, l) = k^{\alpha} l^{1-\alpha}$, where $\alpha = 0.4$. Capital's depreciation rate is $\delta = 0.0196$. Following Guerrieri and Iacoviello (2015), the parameterization of $\phi = 0.975$ implies that the investment constraint is binding about 40% of the time. The TFP shock process takes the form, $z_t = \rho z_{t-1} + \varepsilon_t$, where $\varepsilon_t \sim N(0, \sigma^2)$. The

[11] Christopher Sim's csolve function in MATLAB provides the same results at the expense of greater computation time.

first-order autocorrelation is set at $\rho = 0.95$ and the volatility at $\sigma = 0.007$. The TFP process is discretized into 9 states, as in Rouwenhorst (1995).

We follow the standard procedure in the literature (cf. Judd (1992) and Barillas and Fernandez-Villaverde (2007)) to assess the accuracy of our solutions and calculate the normalized intertemporal Euler equation errors (denoted by EE) implied by the decision rules. The maximum and mean Euler equation errors from the simulation of the model are reported for $10,000$ periods. The results are obtained with MATLAB version R2018a on a PC with Intel(R) Core(TM) i7-7700 CPU 3.60 GHz (3.60 GHz). We use linearly-spaced grids and consider the performance of both methods under a range of different number of grid points with $k_1 = 0.3k^{ss}$ and $k_M = 1.8k^{ss}$, where k^{ss} is the steady-state level of capital. Identical grids are used when comparing the two methods. Baseline calibration yields $k^{ss} = 23.1$ and $i^{ss} = 0.45$.

As shown in Table 1, the main advantage of the GTI algorithm is its speed. The accuracy remains robust across all grid sizes as GTI involves mostly algebraic operations and nonlinear solvers. When the number of nodes is 500, convergence is achieved in 345 iterations and 28.8 CPU seconds. The mean and maximum intertemporal Euler equation errors (EE) are -3.78 and -3.31, respectively (in log 10 units). For PFI, other results seem rather mixed. The method produces relatively large max. Euler equation errors, but this is in part because for comparability the solution is found on an identical and linearly-spaced grid which may be too coarse to accurately describe the policy function (particularly so in the presence of kinks in the policy function). Accuracy improvements require finer grid points, which comes at higher time costs. In general, GTI dominates PFI mostly in terms of speed.

Table 1: Results

Grid points	CPU time	Mean EE error	Max EE error	Iterations
GTI				
10	$2.4s$	-3.72	-3.29	342
500	$28.8s$	-3.78	-3.31	345
$1,000$	$57.0s$	-3.78	-3.38	345
$2,000$	$1m\ 47.5s$	-3.79	-3.29	345
PFI				
10	$.1s$	-2.80	-2.53	2
500	$3m\ 28.5s$	-3.59	-1.22	11
$1,000$	$26m\ 4s$	-3.85	-1.43	11
$2,000$	$3h\ 35m\ 13.3s$	-3.98	-1.40	11

Note: We report the mean and maximum of absolute Euler equation errors (in log 10 units). Errors are obtained from a stochastic simulation of 10,000 periods.

The GTI algorithm requires more iterations for convergence. However, the total time spent for the solution of the problem shows that each iteration is completed faster compared to the time spent for each iteration in PFI. The speed in GTI can be attributed to (i) the time convention of grid points such that the grids are defined in terms of tomorrow's capital rather than today's capital (ii) the solution of capital and labor for the current period is made only once, and only at the end of the algorithm when convergence is achieved. Figures A1 and A2 in the online appendix plot the policy functions for GTI and PFI, respectively.

5 Robustness Analysis

Here we compare GTI and PFI under different scenarios. The baseline scenario considers the benchmark parameterization as described above. In the remaining scenarios, we change one parameter at a time and keep all other parameters at their baseline values. All experiments consider 500 grid points for capital and 9 grid points for productivity shocks. Table 2 (Table 3) summarizes the results for GTI (PFI).

These results confirm that the major advantage of GTI appears to be speed as can be seen from various experiments. GTI helps compute non-smooth decision rules more accurately than PFI, and in scenarios where the constraint is more likely to bind (e.g. $\phi = 1$), it yields more accurate results than PFI.

Table 2: Robustness analysis for GTI

500x9 grid points				
Experiments	CPU time	Mean EE1 error	Max EE error	Iterations
baseline	28.8s	−3.78	−3.31	345
β= 0.96	15.4s	−3.51	−2.91	161
β= 0.99	29.2s	−3.79	−3.30	351
ρ= 0.99	28.3s	−3.69	−2.94	338
ρ= 0.90	28.8s	−3.85	−3.49	351
σ= 0.013	27.2s	−3.55	−3.05	327
α= 0.3	24.8s	−3.63	−3.07	287
α= 0.5	28.5s	−3.97	−3.48	386
ϕ= 0	50.1s	−3.19	−2.96	603
ϕ= 1	23.1s	−3.80	−3.36	269

Note: We report the mean and maximum of absolute Euler equation errors (in log 10 units). Errors are obtained from a stochastic simulation of 10,000 periods.

Table 3: Robustness analysis for PFI

500x9 grid points				
Experiments	CPU time	Mean EE error	Max EE error	Iterations
baseline	3m 28.5s	−3.59	−1.22	11
β= 0.96	3m 5.8s	−3.66	−1.58	10
β= 0.99	3m 14.1s	−3.56	−1.32	11
ρ= 0.99	6m 50.6s	−3.48	−1.17	24
ρ= 0.90	2m 59.4s	−3.15	−2.70	10
σ= 0.013	3m 46.7s	−3.55	−1.10	13
α= 0.3	3m 22.7s	−3.49	−1.26	11
α= 0.5	3m 5.5s	−3.36	−1.27	11
ϕ= 0	4m 16.3s	−3.65	−3.02	15
ϕ= 1	3m 17.2s	−3.16	−1.37	11

Note: We report the mean and maximum of absolute Euler equation errors (in log 10 units). Errors are obtained from a stochastic simulation of 10,000 periods.

In addition to the simulation results, it is possible to see the range of Euler equation errors in Figure 1, where we depict the intertemporal Euler equation errors (in log 10 units) for each method, over 500 capital grid points and 9 productivity shock nodes.

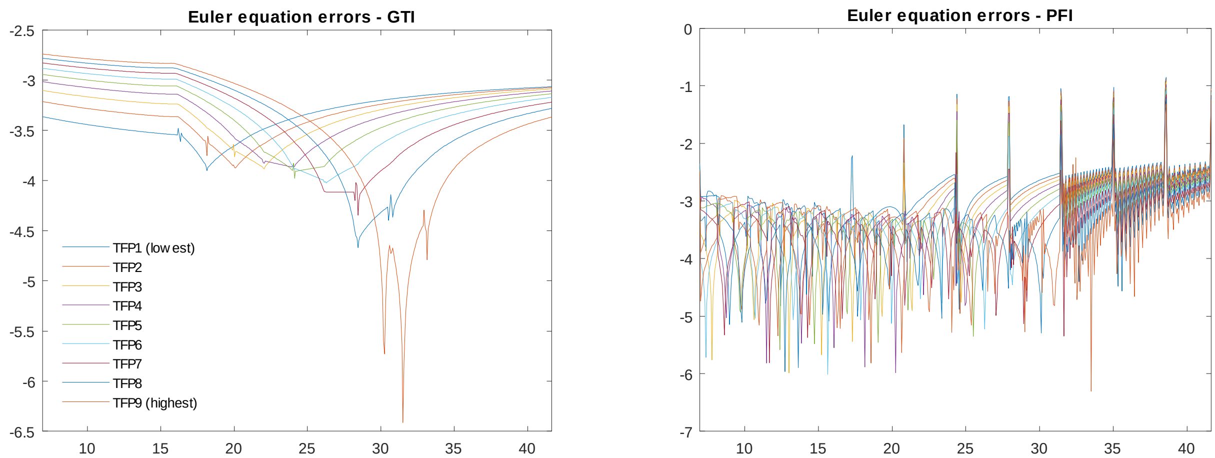

Figure 1: Intertemporal Euler equation errors (in log 10 units) from 500 capital grid points and 9 productivity nodes undert the benchmark calibration.

6 Application: A Two-Country Heterogeneous Agent-Incomplete Markets Model with Progressive Taxes

In this section, we consider an application of the GTI method in a richer framework: a two-country heterogeneous-agent incomplete markets model with flat-rate capital gains taxes, progressive labor income taxes, and government debt. The model can be useful for studying several questions including topics of financial globalization and wealth inequality led by Mendoza et al. (2007) and Mendoza et al. (2009) or redistributive effects of taxation, including the work of Domeij and Heathcote (2004) and Heathcote et al. (2017) among many others.

At the core of the model lies an Aiyagari (1994)-type production economy with uninsured idiosyncratic labor income risk and borrowing constraints. Considering a role for government and the open-economy aspects, the model comes closest to the one in Kabukcuoglu (2017), studying the redistributive effects of tax reform in an open-economy framework. We extend this model with households' labor-leisure choice, applying the GTI method for the solution of households' decision rules. Following Aiyagari (1994) and Judd et al. (2017), the equilibrium interest rate is found using the bisection method.

6.1 The Model

The world economy consists of two countries, Country 1 and Country 2. For convenience, we present the model for a given country, suppressing the country index i. Throughout the model, household variables are denoted by lowercase letters whereas country-level (aggregate) variables are denoted by uppercase letters.

6.1.1 Households

In each country, households are subject to labor productivity shocks, $\varepsilon_t \in E$ which are i.i.d. across households and persistent over time. This is the only uncertainty in the model. In any given period, households make decisions upon the realization of their productivity shock. Household productivity ε_t is assumed to follow a Markov process captured by an $m \times m$ transition probability matrix $\Pi = [\pi_{ij}]$, where $\pi_{ij} = \Pr(\varepsilon_{t+1} = \varepsilon_j | \varepsilon_t = \varepsilon_i)$. The probability distribution over E is given by $p_t \in \mathbb{R}^m$. Given an initial distribution, p_0 the period-t distribution is given by $p_t = p_0 \Pi^t$.

Households' preferences are given by

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t [u(c_t) - g(n_t)] \right] \tag{20}$$

where u is strictly increasing and concave, and g is a strictly increasing and convex function of c_t and n_t, respectively. $\beta \in (0, 1)$ is the discount rate. In each period, a household's consumption is denoted by c_t and hours worked by n_t. We assume identical preferences and labor income processes for both countries.

The pre-tax labor income to a household is given by $\varepsilon_t n_t w_t$, where the w_t is the wage rate. Households' budget constraint can be written as,

$$c_t + \underbrace{b_{t+1} + d_{t+1} + k_{t+1}}_{\equiv a_{t+1}} \leq \tau^n (\varepsilon_t n_t w_t)^{\psi} + \underbrace{[1 + (r_t^k - \delta)(1 - \tau^k)]k_t + [1 + r_t^d (1 - \tau^k)]d_t + (1 + r_t)b_t}_{\equiv (1+r_t)a_t}.$$

A household spends on consumption goods c_t and invest in assets, a_{t+1}. Following Kabukcuoglu (2017) we define three assets with a one-period, risk-free, real return: private bonds, b_{t+1} that pays the world interest rate r_t, government bonds d_{t+1} with an interest rate r_t^d and capital goods, k_{t+1} with a rental rate r_t^k. Capital depreciates at a rate $\delta \in [0, 1]$. The only internationally traded asset is the private bond. Households' after-tax labor income is given by $\tau^n (\varepsilon_t n_t w_t)^{\psi}$ with a two-parameter progressive labor income tax function as in Heathcote et al. (2017). This tax function in general can be written as $\tau(y) = y - \tau^n y^{1-\psi}$ where $y = \varepsilon n w$ denotes pre-tax labor earnings, ψ is a progressivity measure of the tax system (e.g. $\psi = 0$ implies a flat-rate tax). The second parameter, τ^n, is associated with the average taxation of labor income. The tax on capital income is assumed to be flat rate, $\tau^k \in [0, 1]$, implying an after-tax return of $(r_t^k - \delta)(1 - \tau^k)$. Similarly, the government bond's net return is given by $r_t^d (1 - \tau^k)$. Both countries have an identical tax system, while the tax parameters may differ.[12] We abstract from taxes on internationally traded bonds in the current framework.

The no-arbitrage condition is given by,

$$r_t = r_t^d (1 - \tau^k) = (r_t^k - \delta)(1 - \tau^k). \tag{21}$$

Given that households are indifferent as to which asset to purchase in equilibrium, it is possible to state the

[12] See Kabukcuoglu (2017) for a discussion on the possibility of cross-country differences in taxation in a two-country model and the underlying assumptions needed for this result.

household's problem without considering the portfolio composition of assets. The budget constraint can be written as,

$$c_t + a_{t+1} \leq \tau^n(\varepsilon_t n_t w_t)^{1-\psi} + (1+r_t)a_t. \tag{22}$$

Households are also borrowing constrained, determined by an exogenous borrowing limit, $\underline{a}$:

$$a_t \geq \underline{a}. \tag{23}$$

The possibility of having a series of low productivity shocks and the presence of the borrowing limit induces a precautionary savings motive, as in the Aiyagari (1994) framework. It is possible to consider economies with no borrowing, $\underline{a}= 0$, in which case any short position in an asset, e.g. private bonds, must be matched with a long position of the same amount in another asset, e.g. government debt.

For the household problem, it is possible to consider two state variables, (a_t, ε_t) at any period t. Notice that each household's consumption, saving or labor-leisure decision throughout their lifetime will be heterogeneous and depend on the history of the idiosyncratic shocks and their initial conditions (a_0, ε_0). The dynamic optimization problem is described as follows. Given the deterministic sequences of prices $\{w_t, r_t^k, r_t^d, r_t\}_{\tau=0}^{\infty}$, government policy $\{\tau, \tau^k\}$ and initial conditions (a_0, ε_0), a household in a given country chooses a_{t+1}, c_t, and n_t to maximize (20), subject to (22) and (23).

6.1.2 Firms

In each country, output Y_t is produced by a representative firm using aggregate capital K_t and labor N_t according to a constant returns to scale production function, $Y_t = F(K_t, N_t)$. Taking prices (w_t, r_t^k) as given, firm's problem in each country is to maximize profits

$$F(K_t, N_t) - r_t^k K_t - w_t N_t \tag{24}$$

choosing factors both of which are internationally immobile. Perfectly competitive factor markets lead to zero profits in equilibrium.

6.1.3 Governments

Governments can raise revenues by issuing bonds, D_{t+1} at an interest rate, r_t^d, and collecting taxes from households to finance a constant amount of government expenditures, G. The assumption of no international mobility of government bonds allows us to pin down the aggregate amount of private B_t and public bonds D_t in equilibrium. We denote the aggregate tax revenues from progressive labor income taxes $\tau(\cdot)$ by, TR_t. With taxes on capital income and given an initial bond holding D_0, the period government budget constraint can be written as,

$$G + r_t^d D_t = D_{t+1} - D_t + TR_t + K_t(r_t^k - \delta)\tau^k + D_t r_t^d \tau^k \tag{25}$$

6.2 Equilibrium Characterization and Solution

Define A as the set of all possible (household) endogenous states, $A = [\underline{a}, \infty]$. Consider $(A, \mathcal{A})$ and $(E, \mathcal{E})$ the measurable spaces where $\mathcal{A}$ denotes the Borel set that are subsets of A and $\mathcal{E}$ is the set of all subsets of E. Then let $(S, \mathcal{S}) = (A \times E, \mathcal{A} \times \mathcal{E})$ be the product space where S is the set of all possible household states. The solution to the household's problem in Country 1 (and similarly in Country 2) provides the decision rules for consumption, $c_t = c(a_t, \varepsilon_t)$, labor $n_t = n(a_t, \varepsilon_t)$ and asset holdings, $a_{t+1} = s(a_t, \varepsilon_t)$ given the initial conditions (a_0, ε_0) and the history of shocks summarized by ε^t. These rules determine the evolution of the distribution of agents over (a_t, ε_t). The joint (endogenous) distribution of households across wealth and labor efficiency is given by $\mu_t = \mu(a_t, \varepsilon_t)$. A household with the state (a_t, ε_t) has a state vector in the set $A_{t+1} \times E_{t+1}$ next period, given the current distribution and the decision rules. Starting with an initial distribution $\mu(a_0, \varepsilon_0)$, households' distribution across wealth and productivity levels evolve according to:

$$\mu(a_{t+1}, \varepsilon_{t+1}) = \sum_{\varepsilon_{t+1} \in E} \Pi(\varepsilon_{t+1} | \varepsilon_t) \mu(a_t, \varepsilon_t). \tag{26}$$

A general equilibrium under financial integration is characterized by the following conditions.

1. Household Euler equation

$$u_c(c_t^i) = \beta E_{\varepsilon_{t+1}|\varepsilon_t} (1 + r_{t+1}) [u_c(c_{t+1}^i) + \tilde{\lambda}_{t+1}^i], \tag{27}$$

2. Household borrowing constraints, with the associated multiplier, $\tilde{\lambda}_t^i$

$$a_t \geq \underline{a}^i, \tag{28}$$

3. Household intratemporal FOC

$$u_c(c_t^i) \tau'(\varepsilon_t n_t^i w_t^i) \varepsilon_t w_t^i = u_n(n_t^i), \tag{29}$$

4. Aggregations

$$\int_{(a,\varepsilon)} c_t^i d\mu_t^i = C_t^i, \int_{(a,\varepsilon)} n_t^i \varepsilon_t d\mu_t^i = N_t^i, \int_{(a,\varepsilon)} a_{t+1}^i d\mu_t^i = A_{t+1}^i, \tag{30}$$

5. Firm optimization and factor prices

$$r_t^{ki} = F_K(K_t^i, N_t^i), \tag{31}$$

$$w_t^i = F_N(K_t^i, N_t^i), \tag{32}$$

6. Asset market clearing condition

$$A_t^1 + A_t^2 = K_t^1 + K_t^2 + D_t^1 + D_t^2, \text{ for all } t, \tag{33}$$

7. Government budget constraint

$$G^i + r_t^{di} D_t^i = D_{t+1}^i - D_t^i + TR_t^i + K_t^i (r_t^{ki} - \delta)\tau^{ki} + D_t^i r_t^{di} \tau^{ki}, \tag{34}$$

8. No-arbitrage

$$(r_t^{ki} - \delta)(1 - \tau^{ki}) = F_K(K_t^i, N_t^i)(1 - \tau^{ki}) = r_t. \tag{35}$$

The sequence of distributions $\{\mu_t^i\}_{t=1}^{\infty}$ is consistent with the initial distributions μ_0^i, individual policies and idiosyncratic shocks as given by (26) and asset holding positions, A_0^i, D_0^i, K_0^i for all countries $i = 1, 2$.

In addition, it is possible to define investment $I_t \equiv K_{t+1} - (1-\delta)K_t$, net foreign assets $B_t \equiv A_t - K_t - D_t$, current account, $CA_t \equiv B_{t+1} - B_t$, net exports, $NX_t \equiv B_{t+1} - B_t(1 + r_t)$ and net factor payments, $NFP_t \equiv r_t B_t$ based on these aggregates.

6.3 Calibration and Numerical Solution

Focusing on the steady state equilibrium, the solution of the problem involves two steps (i) an algorithm that solves for the equilibrium prices and aggregate variables and (ii) the solution for households' decision rules.

We use $6,000$ equally-spaced asset grid points for policy functions and $120,000$ asset grid points for ergodic distributions. The asset grid has a minimum value of -2 and a maximum value of 200.

Preferences and technology: We specify preferences as $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$, $g(n_t) = B\frac{n_t^{1+1/\eta}}{1+1/\eta}$, and technology as $Y_t = F(K_t, N_t) = ZK_t^{\alpha}N_t^{1-\alpha}$. We let $\gamma = 2, \eta = 0.5$ following Domeij and Flodén (2006) and set $B = 60$ which yields an average hours worked of 32% and 27% of the time endowment in each country, respectively. For the rest of the model, our calibration strategy is in line with Kabukcuoglu (2017), where Country 1 and Country 2 parameters aim to match the US and rest of the OECD data (subject to availability). Capital's share α is set as 0.36, and the depreciation rate δ is 0.06 for both countries. The discount rate β is 0.971. Each country has a unit mass of population. We normalize the TFP parameter $Z^2 = 1$ and set $Z^1 = 0.6$ to generate a realistic GDP share for the U.S., which is 34% in the model.

Borrowing limits: For each country we set $\underline{a} = 0$.

Labor earnings process: We consider a three-state Markov process, following the parameterization of Domeij and Heathcote (2004). We set $E = \{\varepsilon^h, \varepsilon^m, \varepsilon^l\}$ with $\varepsilon^h = 4.74$, $\varepsilon^m = 0.847$, and $\varepsilon^l = 0.170$, in both countries. The transition probabilities are given by

$$\Pi = \begin{bmatrix} \Pi_{11} & 1-\Pi_{11} & 0 \\ \frac{1-\Pi_{22}}{2} & \Pi_{22} & \frac{1-\Pi_{22}}{2} \\ 0 & \Pi_{11} & 1-\Pi_{11} \end{bmatrix} = \begin{bmatrix} 0.90 & 0.10 & 0 \\ 0.005 & 0.99 & 0.005 \\ 0 & 0.10 & 0.90 \end{bmatrix}.$$

The implied steady-state probability distribution is $p^* = [0.0455\ 0.9091\ 0.0455]$.

Government policy: The progressivity parameters are $\psi^1 = 0.151$ and $\psi^2 = 0.151$. We set labor and capital income taxes $\tau^{n1} = 0.73$, $\tau^{n2} = 0.67$, $\tau^{k1} = 0.397$ and $\tau^{k1} = 0.425$, respectively. Initial public debt-to GDP ratios D_0^1/Y_0^1 and D_0^2/Y_0^2 are 0.70 and 0.94, respectively. The steady state government budget implies that the government spending is determined by the calibration of the level of public debt, which therefore, may not match the data closely.

6.4 Results

Steady-state equilibrium prices, allocations, and wealth distributions: The calibration of the model results in a steady-state equilibrium world interest rate of 2.51%; capital-to-GDP ratio of 3.54 (Country 1) and 3.47 (Country 2). The wage rates are given by 0.59 and 1.29, respectively. The resulting government spending to GDP ratios are 51.4% (Country 1) and 73.6% (Country 2). Aggregate tax revenues are 47.3% (Country 1) and 68% (Country 2) of GDP. Finally, we obtain trade balances −1.69% and 0.81% of GDP, respectively, which are not targeted by any of the model parameters. The Gini coefficients for wealth are both 0.56 in the model economies. The policy functions for asset holdings are defined as current period (endogenous) asset holdings over next period's asset grids (and productivity levels) due to the use of endogeneous gridpoints. In a final step, we use linear interpolation to switch to a more conventional definition of policy functions and find next period's asset holdings over (currently-defined) asset grid points. While this is not necessary in computing the equilibrium, it is considered here for presentation purposes. Figure 2 plots these policy functions.

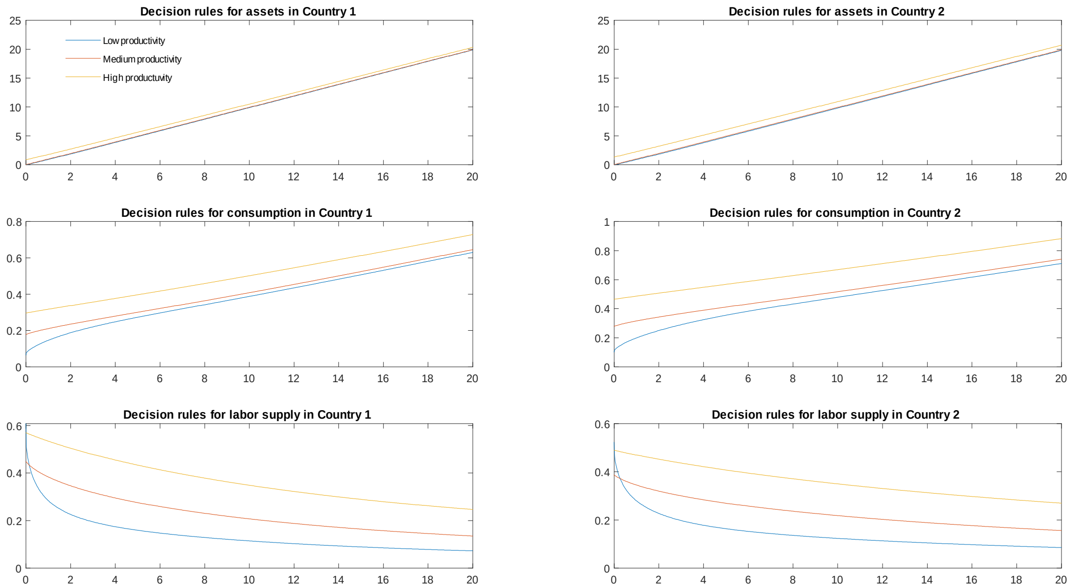

Figure 2: Decision rules for the two-country heterogeneous agent-incomplete markets model computed using 3 productivity shock nodes, 6,000 grid points for decision rules, 120,000 grid points for asset distributions.

GTI performance: We report the maximum and mean intertemporal Euler equation errors from a simulation of $10,000$ periods conducted for each country. For the current model, the Euler equation errors are computed for the cases where the borrowing constraint is not binding. Accordingly, the CPU time for the computation of steady-steady state equilibrium is 43m. 37.5s., where policy functions and ergodic distributions are computed several times for each country over 148 iterations spending approximately 17.6 seconds on average per iteration. The bisection method is known to be sensitive to the choice of the initial guess on the real interest rate and the total computation time varies significantly because of that. In the two-country model, the initial guess on the (world) real interest rate lies between the autarky interest rates of the respective economies, which is below $1/\beta - 1$. The unit-free (absolute) mean and maximum Euler equation errors are very close for the two countries, -3.79 and -1.61 (in log 10 units), respectively. Figure 3 plots the intertemporal Euler equation errors (in log10 units) across 6,000 asset grid points and 3 productivity shock nodes.

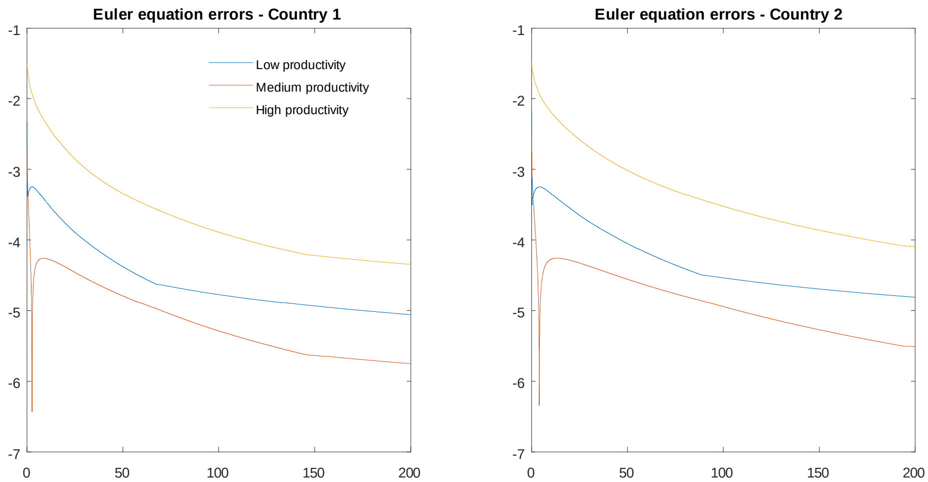

Figure 3: Unit-free absolute intertemporal Euler equation errors (in log 10 units) from 6,000 asset grid points and 3 productivity nodes for each country.

The algorithm for the solution of this problem is presented in an online appendix to this paper.

7 Conclusion

The presence of multiple control variables in a dynamic programing problem may complicate the procedure that is used to find the solution, which may affect the computational time and/or accuracy. Such cases arise in models where there is an endogenous labor-leisure choice in the well-known irreversible investment model. In this paper, we evaluate a generalized version of time iteration (GTI) and show that it yields similar accuracy results to standard policy function iteration (PFI) and outperforms PFI in terms of speed. The applicability and performance of GTI are shown further in a richer heterogeneous agent-incomplete markets model.

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8 Online Appendix

8.1 Figures

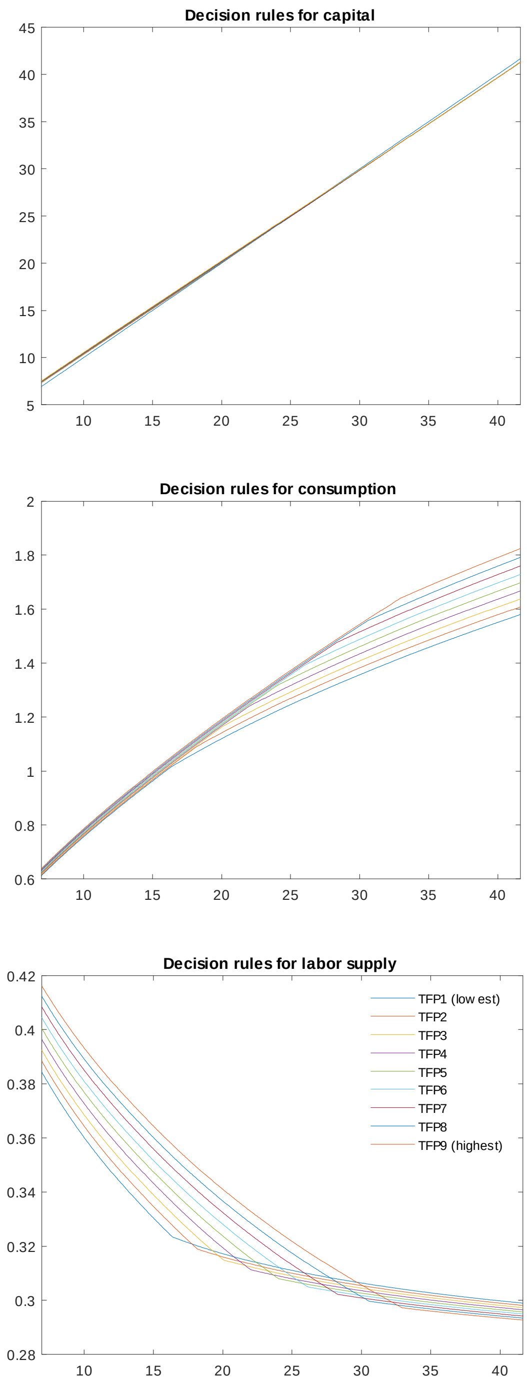

Figure A1: Decision rules obtained under GTI with 500 capital grid points and 9 productivity nodes.

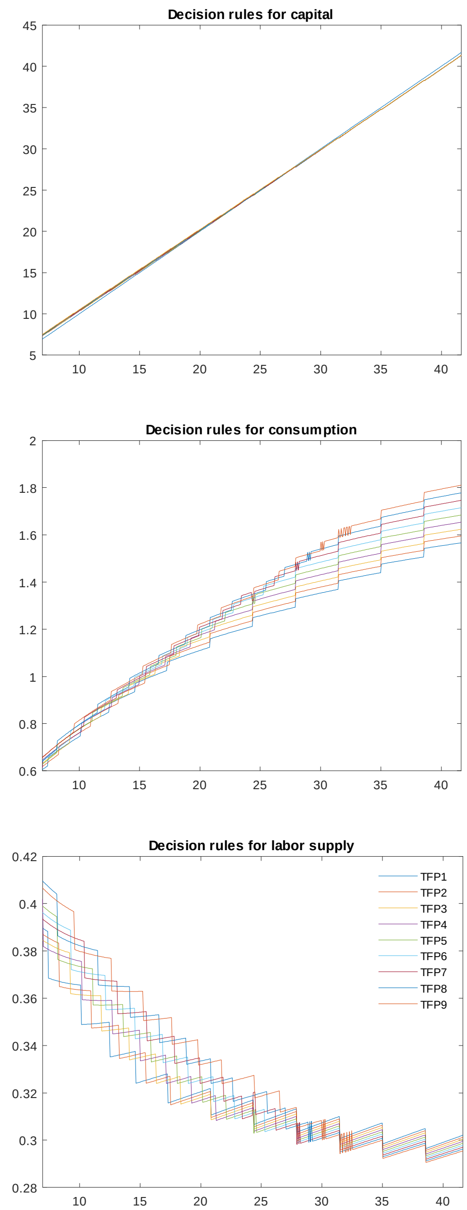

Figure A2: Decision rules obtained under PFI with 500 capital grid points and 9 productivity nodes.

8.2 Solving the Heterogeneous Agent-Incomplete Markets Model with GTI

For each country, generate grid points on next period's assets and current period shocks, (a', ε), where $a' \epsilon A = \{a_1, a_2, ..., a_N\}$. The borrowing limit can be selected slightly above a_1. The borrowing limit is occasionally binding while the upper limit of asset grid points a_N never binds. Define $\varepsilon \in E = \{\varepsilon_1, ..., \varepsilon_M\}$.

1. The interest rate is to be found based on bisection where we start with a guess on r, where $r \in (0, 1/\beta - 1)$. Also set values for D_0 and D_0^*. From the no-arbitrage condition (35), compute the implied K/N ratio and remaining factor prices for both countries: r^{k1}, r^{k2}, w^1, and w^2.

2. Initialize the cumulative distributions of households over assets and shocks, $\Gamma_0(a', \varepsilon)$ for each country.

3. Initialize next period's consumption policy functions, $c_0(a', \varepsilon')$ and do steps 4-9 for each country:

4. Assuming that the constraint is not binding, construct the right hand side of the Euler equation for all pairs of $(a', \varepsilon') \; \epsilon \; A \times E$, and solve for current consumption function $\tilde{c}$,

$$U_c(\tilde{c}) = \beta(1+r) \sum_{\varepsilon' \epsilon E} \Pi(\varepsilon'|\varepsilon) U_c(c_0(a', \varepsilon')).$$

5. Given $\tilde{c}$, using the intratemporal FOC (29), solve for $\tilde{n}(a', \varepsilon)$. The utility function in our example yields an easy computation of the household labor supply, while in other cases the Newton method be needed.

6. Using the budget constraint, compute current asset holdings $\tilde{a}(a', \varepsilon)$ such that

$$\tilde{a}(a', \varepsilon) = [\tilde{c} + a' - \tau^n (\tilde{n} w \varepsilon)^{\psi}]/(1+r) \tag{36}$$

Hence, we find current assets given next period asset holdings is a' and today's productivity shock is ε. Again, the current state $\tilde{a}(a', \varepsilon)$ is not necessarily on the grids defined in A. We then consider two cases:

a. If $\tilde{a}(a', \varepsilon)$ causes the borrowing constraint to bind next period, we compute $\tilde{c}_0(a', \varepsilon)$ using piecewise linear interpolation on the closest grid points a_i and a_j such that, $a_i < \bar{a}(a', \varepsilon) < a_j$ and using consumption rules at $c_0(a_i, \varepsilon)$ and $c_0(a_j, \varepsilon)$. The corresponding labor supply values can be computed from (29).

b. If $\tilde{a}(a', \varepsilon)$ causes the borrowing constraint not to bind next period, then set $\tilde{c}_0(a', \varepsilon) = \tilde{c}$ from step 6.

7. Check convergence for a small value of ε, based on the metric

$$\max\{|\tilde{c}_0(a', \varepsilon) - c_0(a', \varepsilon)|\} < \varepsilon$$

Iterate using steps $4 - 6$ until convergence.

8. Given the initial guess for distributions, $\Gamma_0(a', \varepsilon)$, interpolate on grid points a_i and a_j to find the distribution over the endogenous grid points, $\Gamma(s^{-1}(a', \varepsilon)), \varepsilon)$ using endogenous grids $\tilde{a}(a', \varepsilon)$. Then update the distribution using

$$\tilde{\Gamma}(a', \varepsilon') = \sum_{\varepsilon} \Pi(\varepsilon'|\varepsilon) \Gamma_0(\bar{a}(a', \varepsilon), \varepsilon),$$

and iterate until convergence.

9. Compute aggregate savings, labor, capital and the output level for each country. Find the implied public debt level from the public debt-to-GDP ratio D/Y.
10. Check the asset market clearing condition (33). Update the interest rate, r using bisection method. If necessary, damping may be used.
11. Finally, compute the implied government expenditure, G from (34).